Outline

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   - Main properties of TIs: adiabatic principle and Bulk-edge correspondence
   - Types of TIs at T=0
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   - Symmetry constrains on matrices
3. Su-Schrieffer-Heeger model (1D)
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4. Chern insulator and Quantum spin Hall (2D)
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   - Spin-orbit coupling
   - Stability of QSH and edge states
5. Time-reversal invariant Topological insulator (3D)
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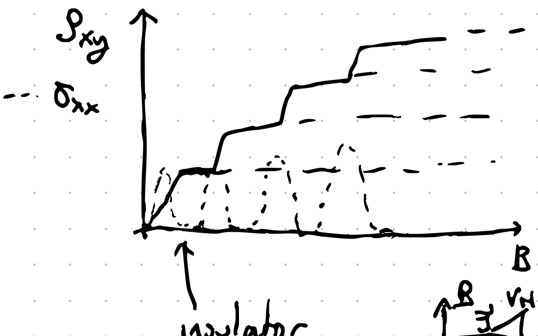
6. Exercises:
   - Warm up: left-over manipulations
   - Explicit model for a quantum spin Hall
   - Constructing a Weyl semimetal

Ref: https://grushingroup.cnrs.fr/topointro2021/

1] General intro

Imagine you are Klaus von Klitzing and you see:



← all values are quantized

@ plateaus:

$$\sigma_{xx} = 0$$

$$\sigma_{xy} = \nu \frac{e^2}{h}$$

with ν an integer ($\mathbb{Z}$) [WTF]

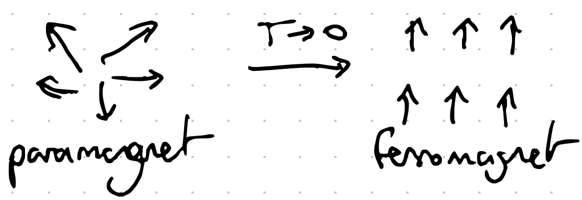
How can you explain this universal response!?

<u>Ask Questions:</u>

1. <u>Symmetries</u>: is there symm. breaking?

No. Seems transitions occur w/o symm. breaking

<u>reminder</u>



local order parameter $\langle \vec{M}(\vec{x}) \rangle = 0 \longrightarrow \langle \vec{M}(\vec{x}) \rangle \neq 0$

Here, nothing of the sort.

<u>ABSENCE</u> of local order parameter

QHE

2. Dimensionality? $d = 1, 2, 3, \ldots$

3. Which particles? bosons, fermions? fermions (e^- current)

4. gap vs no gap? $\sigma_{xx} = 0 \Rightarrow$ insulator

but certainly something is conducting

$$\sigma_{xy} = C \, e^2/h$$

5. Equilibrium? Close (linear response)

$$j_i = \sum_j \sigma_{ij} E_j$$

Far from eq needs other techniques

→ Keldysh, Lindblad eq, Non-Hermitian Hamiltonian.

Many platforms
- Solid state: SOC systems, magnetic, chiral crystals
- Metamaterials: Photonic crystals, Acoustic, mechanical, atomic lattices, polaritonics
- Time dependent (Driven systems)

1.1) Main properties of topological insulators

Naive pic: A frustrating gift

something between metal and insulator:

insulator — $|\psi|^2$ is localized

topological insulator — ?

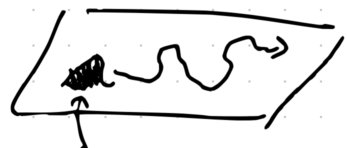
metal — $|\psi|^2$ is delocalized

Folklore

(A) Described by <u>global</u> properties of wave function.

→ Need info of $[\psi_k]$ in all the BZ or $[\psi(\vec{r})]$ in all real space to know if system is topological

⇒ You can get away with less if you have symmetry.



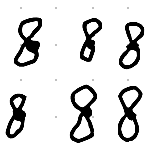
(B) Characterized by quantized responses: observables equal integer (fraction) times fundamental constants

$$\sigma_{xy} = \underset{\text{Chern number}}{C} \, e^2/h$$

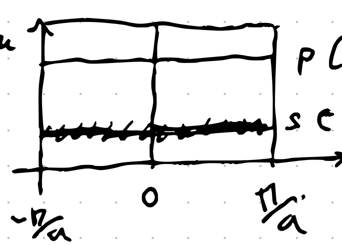
(C) Not <u>connected</u> to <u>atomic insulators</u>
? ?

<u>atomic insulator</u>:

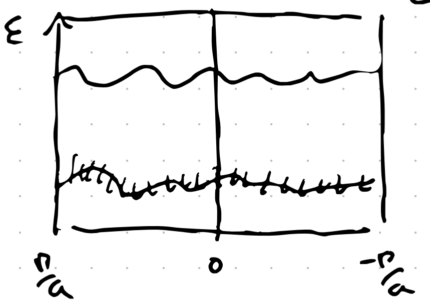
Collection of atomic orbitals without hoppings:



⇔ Flat bands (≡ trivial) insulator

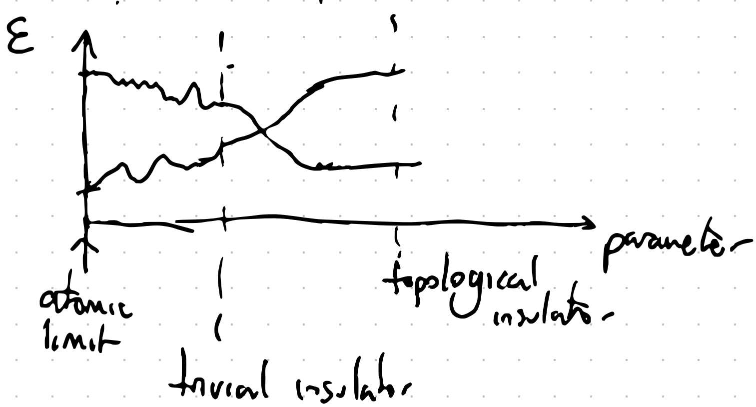


Now turn on slowly hoppings:



Bands start to disperse but gap remains open.
"adiabatically connected" ≡ as long as I turn on hoppings slower than the band gap energy we are connected to the atomic insulator limit. ⇒ "Adiabatic principle"

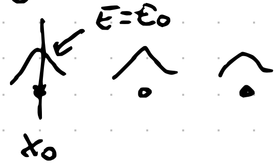
A topological insulator is an insulator that it is not adiabatically connected to an atomic insulator ⇒ a gap closing occurs in the process of varying the parameters of my Hamiltonian.



Implies: a) Non-Wannierizability:

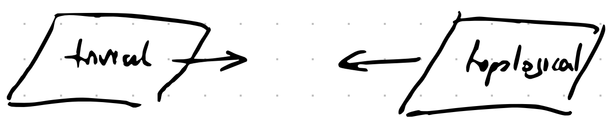
$$[\hat{x}_i, H] \neq 0 \quad \text{for some } \hat{X}_i$$

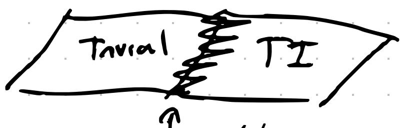
I can't find eigenstates with well defined position and energy:



$\equiv$ trivial

b) edge states





metallic edge-state.

as you move from inside to outside of a TI at some point the gap must close $\equiv$ boundary.

The connection between bulk properties and the presence of edge states is sometimes called the bulk-boundary correspondence.

Types of (T=0) gapped topological phases

Entanglement : Short range (think product state)
Long range (think FQHE)

SRE — Connected to prod state by local unitaries

- Symmetry Breaking
  e.g. Ising FM
- non-symmetry breaking
  - trivial
    (can be symmetry graded)
  - Symmetry protected topological phases
    (e.g. topological insulators)
    (e.g. 1D odd-spin chains)

This lecture :)

LRE — Not connected; anyons, top. ent entropy, deg of GS on torus...

- symmetry enriched topological phases
  (some classes of spin liquids)
- no need for symmetry
  - sym broken states
    (e.g. chiral spin liquids)
  - No symm breaking
    (FQHE)

Insulators with n-bands

Topological insulators

≡ not connected to atomic limit

Crystalline

Protected by unitary symm (e.g. inversion)

Strong

Protected by discrete symm (T, C, S)

Fragile

adding bands to TI turns them trivial

Trivial insulators

≡ connected to atomic limit

Obstructed

symmetry prevents connecting trivial ins w/o band gap closing.

Learning by examples

0D Topological phases

Start with a diatomic molecule with one spinless electron

$\underset{A}{\bullet} \overset{t}{—} \underset{B}{\circ}$ $\quad H = t c_A^\dagger c_B + t c_B^\dagger c_A \Rightarrow h_1 = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$ ← first quantized H

H: second quantized H $\qquad |\psi\rangle = \begin{pmatrix} c_A \\ c_B \end{pmatrix}$

compare it to $\bullet \overset{-t}{—} \circ \Rightarrow h_2 = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix}$

$\underline{h_1}$ $\quad \epsilon_\pm^{(1)} = \pm t \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \equiv |\psi_+\rangle \\ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \equiv |\psi_-\rangle \end{cases}$

E: t — $|\psi_+\rangle$ ← empty; $-t$ — $|\psi_-\rangle$ ← filled

$\underline{h_2}$ $\quad \epsilon_\pm^{(2)} = \pm t \begin{cases} \begin{pmatrix} -1 \\ +1 \end{pmatrix} \text{ "antibonding"} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ "bonding"} \end{cases}$

E: t — $|\psi_-\rangle$ ← empty; $-t$ — $|\psi_+\rangle$ ← filled

- Are these two h_1, h_2 connected by a path in parameter space w/o closing the gap?

$\underline{\text{General } h:}$ (Molecule 1) $\quad h = \underbrace{\varepsilon_0}_{\text{numbers}} \mathbb{1}_{2\times 2} + d_x \sigma_x + d_y \sigma_y + d_z \sigma_z \equiv \varepsilon_0 \sigma_0 + \vec{d} \cdot \vec{\sigma}$

with $\mathbb{1}_{2\times2} \equiv \sigma_0$, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; d_x, d_y, d_z: numbers

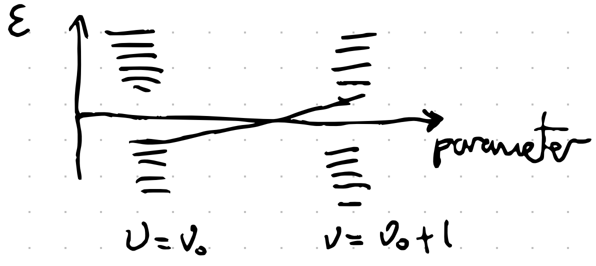
$h_1 \to h_2$ is possible w/o closing the gap. Why?

$\epsilon_\pm = \varepsilon_0 \pm \sqrt{d_x^2 + d_y^2 + d_z^2} \Rightarrow$ as long as $|d| \neq 0$ gap remains open!

Bonus: Invariant is

$$\nu = \tfrac{1}{2}\,\mathrm{sgn}(H) = \#\text{ positive eigenvalues} - \#\text{ negative eigenvalues} \in \mathbb{Z} \;\leftarrow \text{integer}$$

Why? The only time when I cannot connect them is when an eigenvalue crosses zero:



Option 1: Impose inversion

Inversion changes $A \leftrightarrow B$ $\quad \sigma_x \equiv \hat{I}$ = Unitary symmetry

symmetry means $I h I^{-1} = h \Rightarrow \sigma_x h \sigma_x = h$

Is this true?

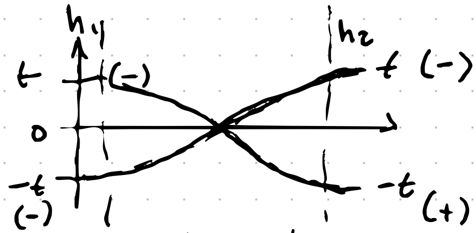
$$\sigma_x (\varepsilon_0 \sigma_0 + d_x \sigma_x + d_y \sigma_y + d_z \sigma_z) \sigma_x \stackrel{?}{=} h$$

$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \Rightarrow$ whenever I exchange two Pauli matrices I pick a -ve sign if they are different

$\Rightarrow \varepsilon\sigma_0$ ✓, $d_x \sigma_x$ ✓ (exercise)

Our original h_1, h_2 have eigenstates of inversion

$\Rightarrow \sigma_x |\psi_\pm\rangle = \pm |\psi_\pm\rangle$



The only path that preserves inversion

Half filling: always a gapless state in the middle

We will see this is not robust in general! disorder can gap out

Notice we can write the hamiltonian in block diagonal form $\quad u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow$

$$u h u = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix}\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -t & t \\ +t & t \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & +1 \end{pmatrix} = \left(\begin{array}{c|c} -t & 0 \\ \hline 0 & t \end{array}\right)$$

each block has a definite inversion eigenvalue

True for all unitary symm:

Option 2: Impose Chiral symmetry and time-reversal symmetry
(Molecule 2)

TRS for spinless systems: $\hat{T} = K$ (complex conjugation)

Why? $T[\hat{x}, \hat{p}]T^{-1} = T i\hbar T^{-1}$ — Not rigorous but correct.

$\hat{x} \xrightarrow{T} \hat{x}$ $\quad \hat{p} \xrightarrow{T} -\hat{p}$ $\quad$ so $\quad i \to -i$

$$T h T^{-1} = h \Rightarrow h^* = h \quad \text{(spinless)}$$

$\Rightarrow$ TRS is Antiunitary sym $T = U_T K$ (unitary x complex conj)

Chiral symm: $S = U_S \Leftrightarrow U_S h U_S = -h$

it is a sort of sublattice symm. it assigns different eigenvalues to different sublattices (A, B)

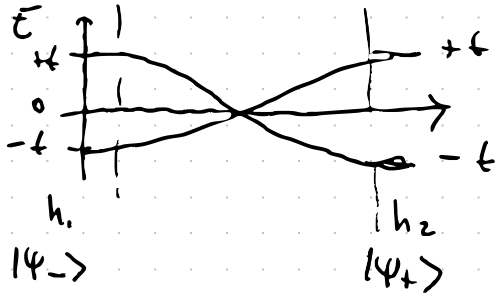
$\Rightarrow U_S = \sigma_z$

Same game: $\quad$ complex so if d's are real $\Rightarrow d_y = 0$ bc of T

1. $\sigma_z(\varepsilon_0\sigma_0 + d_x\sigma_x + \overbrace{d_y\sigma_y}^{\text{complex}} + d_z\sigma_z)\sigma_z = -h$

2. $\sigma_z(\cancel{\varepsilon_0\sigma_0} + d_x\sigma_x + \cancel{d_z\sigma_z})\sigma_z = -h$

$\Rightarrow h = d_x\sigma_x$ is the only possibility:



eigenval are still inversion eigenvalues!!

educated guess:

invariant can be computed using inversion eigenvalue

$\Rightarrow h_1 \to -1$ $\quad$ is this a

$h_2 \to +1$ $\quad$ coincidence?

we will see it is not.

Crystal symm. (e.g. inversion) allow computing invariants easily!

Actual invariant $h = \begin{pmatrix} 0 & C \\ C^\dagger & 0 \end{pmatrix} \Rightarrow \nu = \mathrm{sgn}(\det(C)) \in \mathbb{Z}_2 = 0, 1$

(because of chiral sym and TRS)

Ref 0D: "K-Theory and pseudospectra for topological insulators" T. Loring 1502.03498

What have we learned?

1. Symmetries constrain Hamiltonians
2. Symmetries constrain which Hamiltonians are adiabatically connected w/o gap closings
3. You can calculate "invariants" easily by using unitary symmetries
4. There are 4 types of symmetries to consider:

Bas's Lecture

Block diagonalizes → Weak TI's, HOTIs, ...)

$-k$ for inversion.

Unitary commute $U h_k U = h_{ugh}$ (like inversion)

Unitary anti-comm $U_S h_k U_S = -h_k$ (chiral symm)

Anti unitary commute $U_T h_k^* U_T = h_{-k}$ (time reversal)

Anti-unitary anti-comm $U_e h_k^* U_e = -h_{-k}$ (particle hole)

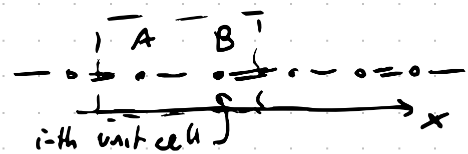
Form the strong topological insulator classes.

Do not depend on momentum (yet)

We will barely use particle-hole these lectures, see lectures by A. Meszaros and T. Chen

1D | Su-Schrieffer-Heeger Chain

Start with Molecule: $h_{SSH} = h(k_x)$



$$H = \sum_i \left[t(1-\delta)\, c_{iA}^\dagger c_{iB} + t(1+\delta)\, c_{iB}^\dagger c_{i+1A} + h.c. \right] +$$

$$+\, m \left[c_{iA}^\dagger c_{iA} - c_{iB}^\dagger c_{iB} \right]$$

$$= \underbrace{H_{SSH}(t,\delta) + H_m}_{\text{Rice-Mele model}}$$

Choose $t=1$ $m=0$

SSH model

$a=1$

$x_i = ia$

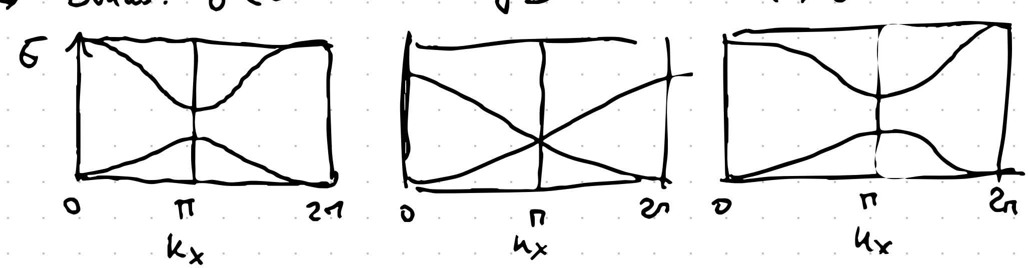
$$\Rightarrow c_{i\alpha} = \frac{1}{\sqrt{L}} \sum_i e^{ikx_i} c_{k\alpha} \qquad \alpha = A, B$$

$$\Rightarrow \text{(exercise)} \quad \vec{d} = \left((1-\delta) + (1+\delta)\cos(ka),\ (1+\delta)\sin(ka),\ 0 \right) \qquad a=1$$

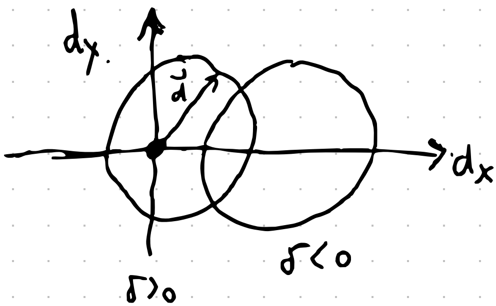
$$h_k^{SSH} = d_x^k \sigma_x + d_y^k \sigma_y \quad \rightarrow \epsilon_\pm = \pm\sqrt{d_x^2 + d_y^2}$$

$$d_x^k = (1-\delta) + (1+\delta)\cos k \qquad d_y^k = \sin k$$

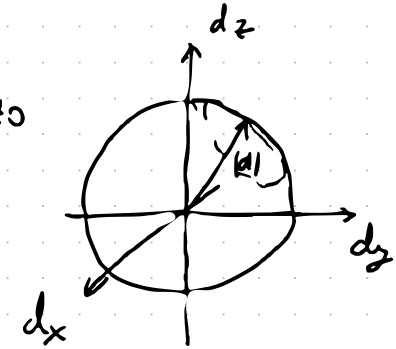
$\Rightarrow$ Bands: $\delta < 0$ $\qquad$ $\delta = 0$ $\qquad$ $\delta > 0$



Are these insulators different?



iff $d_z \neq 0$



$|d| = 0$ is special because gap closes. So $\delta > 0$ and $\delta < 0$ are connected by a gap closing transition but also, wind differently around the origin.

Iff $d_z \neq 0$ all paths become equivalent $\Rightarrow$ we need symmetries to avoid this

- Let's use <u>chiral symmetry</u>:

$$U_S h_k U_S = -h_k \rightarrow \text{exercise} \Rightarrow \left[\begin{array}{l} d_z = -d_z = 0 \\ d^x_k = d^x_k \\ d^y_k = d^y_k \\ \varepsilon_0 = -\varepsilon_0 \end{array}\right.,$$

$$U_S = \sigma_z \qquad U_S^2 = 1$$

This symmetry constrains $\vec{d}$ to lie on the plane

$\Rightarrow$ two types of insulators depending on whether we encircle the plane.

How to construct the invariant, and physical meaning

nth winding around the origin

$$d_x\sigma_x + d_y\sigma_y \Rightarrow \vec{d} = (d_x, d_y) \equiv |d_k|\, e^{i\varphi_k}$$

$$\varphi_k = \arctan\left(\frac{d_x}{d_y}\right)$$

$$\nu = \frac{1}{2\pi}\int_{BZ} dk\, \frac{\partial \varphi_k}{dk} = 0, 1, \ldots \in \mathbb{Z}$$

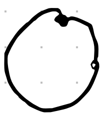
$\equiv$ # of times vector $\vec{d}$ winds around the origin

(homotopy: $\pi_1(S_1) = \mathbb{Z}$)

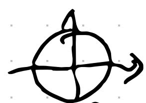
$k \in BZ$ in 1D is a circle

d vector lives in a circle

how many times we cover the circle of d by spanning the circle of k



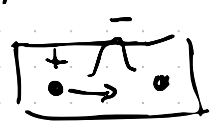
$k \in BZ \equiv S_1$



S_1

Physics: Berry phase, Zak phase, Polarization

Polarization: dipole moment of charge



$\vec{P} \sim \langle \hat{x} \rangle$ of charge center.

$\vec{P} \simeq$ Average displacement of e^- cloud $\sim \langle i\partial_k \rangle$

$$P = \sum_{n \in occ} P_n \quad ; \quad P_n \equiv -\frac{e}{2\pi}\int_0^{2\pi} dk\, \langle u_k^n | i\partial_k | u_k^n \rangle$$

$$P_n \equiv -\frac{e}{2\pi}\int_0^{2\pi} dk \langle u_k^n | i\partial_k | u_k^n \rangle \equiv \text{polarization}$$

$|u_k^n\rangle \equiv$ Periodic part of Bloch function.

$$P_n \equiv \frac{e}{2\pi}\,\Phi_n \equiv \bar{x}_n \leftarrow$$ average position of Wannier center (see notes)

Zak Phase $\equiv \Phi_n$

<u>Zak Phase</u>: a particular instance of Berry phase

$$\gamma_c = \oint_c \langle u_\lambda^n | i\partial_\lambda | u_\lambda^n \rangle \equiv \text{Berry phase}$$

For Zak phase: $C \equiv BZ \qquad \lambda \equiv k$

Berry connection: $\equiv \vec{A}_k^n \equiv \langle u_k^n | i\vec{\partial}_k | u_k^n \rangle$

vector in general D

Transform $|u_k^n\rangle \to e^{i\phi_k^n}|u_k^n\rangle$

then $\vec{A}_k \to \vec{A}_k + \vec{\partial}\phi_k$

then $P_n \to P_n + me$ (exercise) $m \in \mathbb{Z}$

Polarization only defined modulo e!

Only changes in polarization (i.e. currents) are observable! e.g. $\vec{j} = \frac{\partial \vec{P}}{\partial t}$ we will see that.

Polarization for SSH?

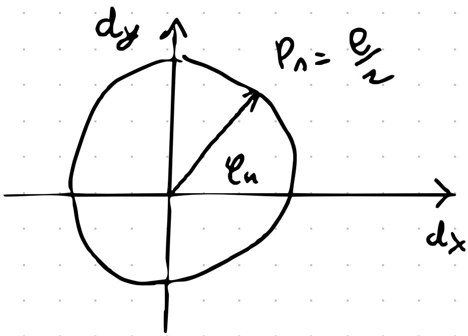
$$h = d_x \sigma_x + d_y \sigma_y \Rightarrow (d_x, d_y) \equiv \vec{d} = |d| e^{i\varphi_k}$$

$$|u_k^\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp e^{i\varphi_k} \\ 1 \end{pmatrix}$$

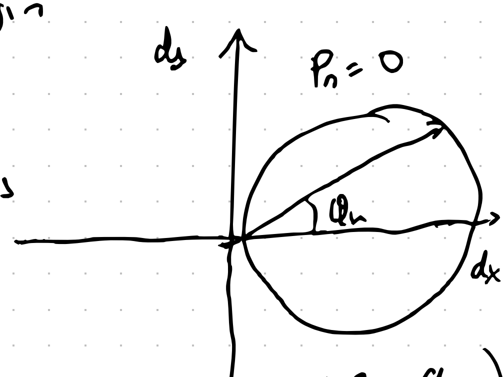
$$P_1 = \frac{1}{2} \frac{e}{2\pi} \int_0^{2\pi} dk \, \frac{\partial \varphi_k}{\partial k} = \frac{me}{2} \mod e$$

$$\phi_k = \phi_{k+G} \mod 2\pi$$

So P_1 is measuring the winding mod $e/2$ of the vector d at the origin



vs



More generally chiral symmetry has $h = \begin{pmatrix} 0 & q_k \\ q_k^+ & 0 \end{pmatrix}$

Invariant:
$$\begin{cases} \nu = \frac{i}{2\pi} \int_k \mathrm{Tr}\left[q_k \partial_k q_k^+ \right] \in \mathbb{Z} \\ P = e\frac{\nu}{2} \mod(e) \end{cases}$$

Do we need to know all winding? Not <u>if</u> we have inversion:

At $k = 0, \pi$ we have the familiar hamiltonian:

$$h(0) = 2\sigma_x \quad \leftarrow d \parallel \hat{x}$$

$$h(\pi) = -2\delta\,\sigma_x \quad \rightarrow d \parallel \hat{x} \text{ or } -\hat{x}$$

δ controls which $\hat{I}$ eigenvalue is occupied

The 0D molecule!

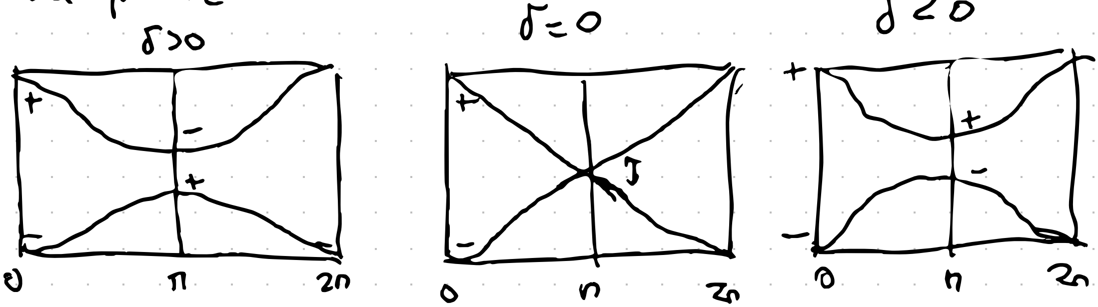
We can simply multiply inversion eigenvalues to find the invariant (mod 2)

$$\xi_{k=0}\,\xi_{k=\pi} = \pm 1 \quad \text{trivial/topological}$$

$\equiv$ Fu-Kane Formula

$\xi_{k=0}$, $\xi_{k=\pi}$: inversion eigenvalue @ $k=0$ or $k=\pi$ because $k=-k$ and $\hat{I} h_k \hat{I} = h_{-k}$

Band picture



$\xi_{k=0}\,\xi_{k=\pi} = -1$ (topological)

$P_n = \frac{e}{2}$ mode

?

$\xi_{k=0}\,\xi_{k=\pi} = +1$ (trivial)

$P_n = 0$ mode

Lessons:

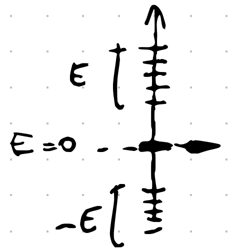
- Unitary Symmetry simplifies calcs of invariants
  BUT we might miss info ($\mathbb{Z}_2$ vs $\mathbb{Z}$)
- Can be enough to know info @ specific momenta/lines/planes (Ben's lecture)
- Invariants have physical meanings ($P \sim$ Zak phase)

Last thing on SSH:

Edge states

Semi-infinite chain 1 • 2 o = • 3 o = • 4 5 6 o = • — — — C ⊙

$c^\dagger_{A1}$, $c^\dagger_{BN}$ do not appear in the Hamiltonian
$\Rightarrow$ zero modes



Chiral symmetry For each $E>0$ you have one at $E<0$

(exercise) $\begin{cases} h|\psi\rangle = \epsilon|\psi\rangle \\ h(S|\psi\rangle) = -Sh|\psi\rangle \\ h(S|\psi\rangle) = -\epsilon(S|\psi\rangle) \end{cases}$

Can't move it up or down: either I break chiral symmetry or I have a very non-local coupling

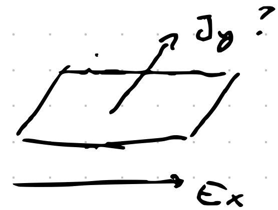
$\hookrightarrow c^\dagger_{A1}c_{A1}$ $\qquad\qquad \hookrightarrow c^\dagger_{A1}c_{BN}$

You can find them by doing an expansion around $k=\pi$ and looking at what happens at the boundary between $\delta>0$ and $\delta<0$

$A - B - A - B \underbrace{\}}_{\text{zero mode}} B \quad A \quad B \quad A$

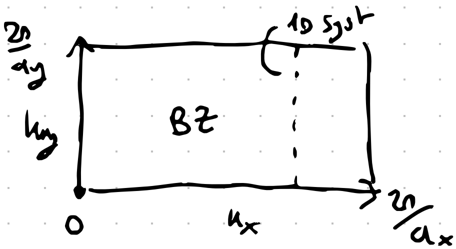
Chern insulators (2D)

$$j_i = \sigma_{ij} E_j \quad ; \quad \sigma_{xy} = C\, e^2/h$$



How?

Start with a 2D-BZ



$$h^{2D} = h(k_x, k_y)$$

Imagine that at each K_x we calculate the Polarization of the corresponding 1D system:

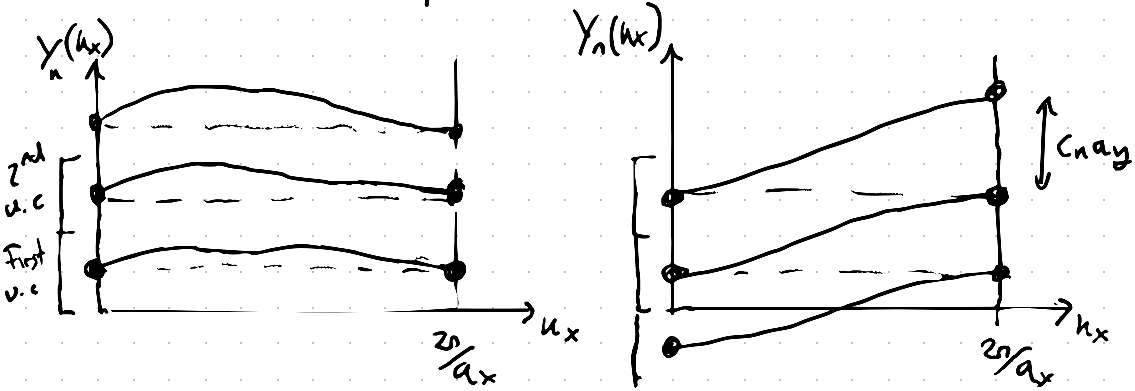
Zak Phase $\quad \Phi_n(k_x) = \int_0^{2\pi/a_y} dk_y \langle u_k^n | i\partial_{k_y} | u_k^n \rangle$

remember Zak phase determines position of Wannier center $\quad \bar{Y}_n(k_x) \equiv \Phi(k_x)$

We want to know how the polarization changes as we move k_x from $0 \to 2\pi/a_x$

define coordinate $\bar{y}_n(k_x) = \frac{a_y}{2\pi} \Phi(k_x)$

There are two options



Both are periodic. Any integer "pump" is allowed

$$\Delta y = \frac{a}{2\pi}\left[\Phi(k_x = 0) - \Phi\left(k_x = \frac{2\pi}{a_y}\right)\right]$$

$= a_y C_n$ where $C_n \in \mathbb{Z}$ Chern number.

Counts how many charges are pumped across unit cells.

Hall conductivity? Newton's law: $\hbar \dot{k}_x = e E_x$

$$\Delta k_x = 2\pi/a_x \Rightarrow \frac{h}{2\pi}\frac{\Delta k_x}{\Delta t} = eE_x \Rightarrow \Delta t = \frac{h}{eE_x a_x}$$

time for momentum to change from $0 \to 2\pi/a_x$

Current density in y direction

$$J_y = ne\langle v\rangle_y = \frac{e}{a_x a_y}\frac{\Delta y_n}{\Delta t} = \frac{C_n \cancel{a_y a_x} e^2 E_x}{\cancel{a_x a_y}\, h} = C_n \frac{e^2}{h} E_x!$$

n: $1e^-$/u.c volume

Quantized Hall effect!

How did this happen?

Applying an $\mathcal{E}_x$ pumped a charge from 0 to $2\pi/a_x$ but because we had a Polarization in the ~~y~~ direction we generated a current.

$$\sigma_{xy}^T = \sum_n C_n \, e^2/h$$

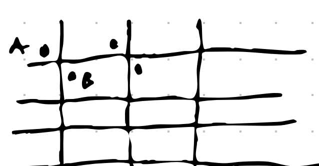
More generally we calculate the total charged pumped by integrating over the 2D BZ the change in polarization:

$$\sigma_{xy} = \frac{e^2}{\hbar} \sum_n \int_{BZ} \frac{d^2k}{(2\pi)^2} \left(\Omega_k^n\right)_z \overbrace{n_F(\epsilon_k^n - \mu)}^{\text{Fermi-Dirac dist fnct.}}$$

$$\vec{\Omega}_k^n \equiv \vec{\partial}_k \times \vec{A}_k^n \equiv \text{Berry curvature} \equiv \text{Mag field in momentum space}$$

It is easy to see why this works with 2 band models:

Chern insulator model



$$H = \sum_i t c_i^\dagger \left(\frac{\sigma_z - i\sigma_x}{2} \right) c_{i+\hat{x}} + h.c.$$
$$+ t c_i^\dagger \left(\frac{\sigma_z - i\sigma_y}{2} \right) c_{i+\hat{y}} + h.c.$$
$$+ M c_i^\dagger \sigma_z c_i$$

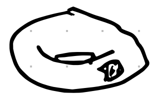
$\Rightarrow$ Fourier space (exercise), $\Rightarrow h = \epsilon_0^k \sigma_0 + \vec{d}_k \cdot \vec{\sigma}$

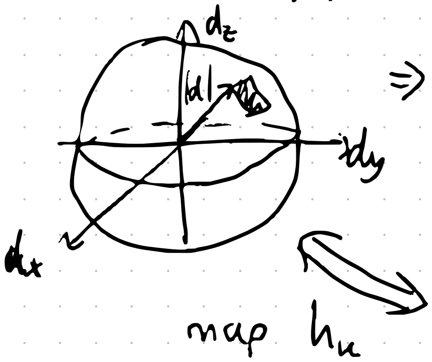
$\epsilon_0 = 0 \qquad \vec{d} = (\sin k_x, \sin k_y, M/t - \cos k_x - \cos k_y)$

Math

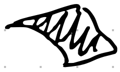
No symmetries imposed $(\cancel{T}, \cancel{C}, \cancel{S})$

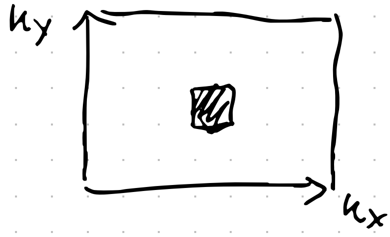
so $\hat{d} = \vec{d}/|d|$ lives on a sphere:





$\Rightarrow$ as we move in $k_x, k_y \in T^2$

we trace patches 



Jacobian of transformation:

$\vec{k} \to \vec{k} + dk_x \hat{x}$

$d_k \to d(k + dk_x \hat{x}) - d_k \simeq \partial_{k_x} \hat{d}_{\vec{k}} \, dk_x$

Same for y:

$$d\vec{S}_k = (\partial_{k_x} \hat{d}_k \, dk_x) \times (\partial_{k_y} \hat{d}_k \, dk_y)$$

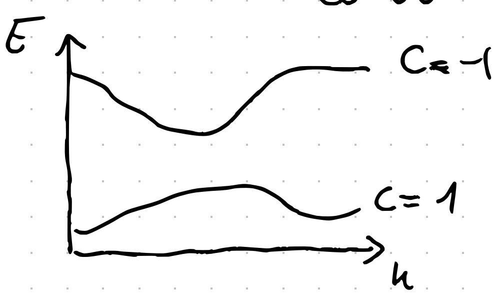
Calculate solid angle:

$$C = \frac{1}{4\pi} \int \hat{d}_k \cdot d\vec{S}_k = \frac{1}{4\pi} \int_{BZ} dk_x dk_y \, \hat{d}_k \cdot (\partial_{k_x} \hat{d}_k \times \partial_{k_y} \hat{d})$$

= skyrmion # over a closed space $\in \mathbb{Z}$

$\hat{d}_k \cdot (\partial_x \hat{d}_k \times \partial_y \hat{d}_k) \equiv \Omega^1_k = -\Omega^2_k$ for a two model:

$$h = \varepsilon_0 \sigma_0 + \vec{d}\vec{\sigma} \quad (\varepsilon_0 \text{ does not enter!})$$



<u>Things:</u>

- Each isolated band has a Chern number $\in \mathbb{Z}$
- TRS $\Leftrightarrow$ $C_n = 0$ $\quad \Omega^n_k = -\Omega^n_{-k}$ (show)
- Inversion $\Leftrightarrow$ $\Omega^n_k = \Omega^n_{-k}$
- I + TRS $\Leftrightarrow$ $\Omega_k = 0$
- $\sum_n C_n = 0$ if n goes over all bands

<u>Chern # via symmetry indicators</u> [Hughes, Prodan, Bernevig PRB <u>83</u>, 245132 (2011)]

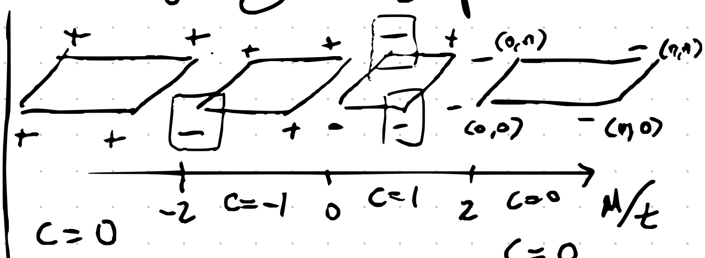
$$\vec{h} = (\sin k_x, \sin k_y, M/t - \cos k_x - \cos k_y)$$

This model has inversion symmetry: $\quad C^n = \prod_{k \in TRIMs} \xi^n_k \mod(2)$

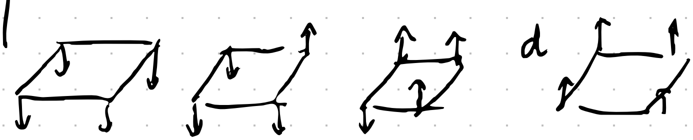
$$I h_k I^{-1} = h_{-k} \quad \text{with } I = \sigma_z$$

Inversion eigenvalues at high symmetry points:

	$\vec{d}$
$(0,0)$	$(M-2)\sigma_z$
$(0,\pi)$	$M\sigma_z$
$(\pi,0)$	$M\sigma_z$
(π,π)	$M+2\ \sigma_z$

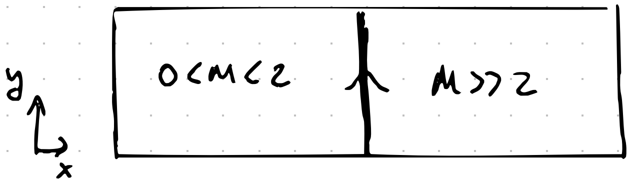


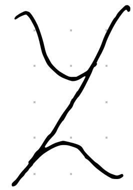
lowest band has − eigenvalue



You can also calculate it explicitly with $\hat{d}$

Edge states:





gap closing at the edge.

TRS is broken $\Rightarrow$ 1 chiral edge state.

Solve Sch. eq with

and you will find only one normalizable solution with $v_y > 0$

Quantum spin Hall : 2D + spinful TRS $\hat{T} = i\sigma_y K$

① A small note on $T^2 = -1$ (spinful) vs $T^2 = +1$ (spinless)

$T^2 = -1$ has Kramers-degeneracy:

Time reversal related states have the same energy:

- $H|\psi\rangle = E|\psi\rangle$
- $H\,T|\psi\rangle = TH|\psi\rangle = E|T\psi\rangle$ same state?

$|\phi\rangle = T|\psi\rangle$
$|\varphi\rangle = |\psi\rangle$

$\langle T\phi|T\varphi\rangle \overset{(1)}{=} \langle\varphi|\phi\rangle$ (action of antiunitary)

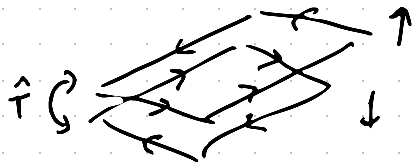
$\Rightarrow \langle T^2\psi|T\psi\rangle = -\langle\psi|T\psi\rangle \underset{(1)}{=} \langle\psi|T\psi\rangle = 0 \Rightarrow \perp$ orthogonal

② SOC: $\vec{L}\cdot\vec{S}$ is TRS because both $\vec{L}$ and $\vec{S}$ are odd under TRS

B field $(p-A)^2 \rightsquigarrow \vec{p}\cdot\vec{A} = B_z L_z$ $\quad p_x A_x + p_y A_y = -p_x y B_z + p_y x B_z = |(\vec{r}\times\vec{p})|_z \cdot B_z$

$A = (-B_z y, B_z x, 0)$

So SOC is like opposite B field for different spins:



Two copies of Chern insulators = QSH effect.

<u>BUT</u>: Can these couple and gap out?

NOT if $T^2 = -1$!

$$\langle T\psi_1 | V_{pert} | T\psi_2 \rangle \overset{\text{if } [H,T]=0}{=} \langle \psi_2 | H | \psi_1 \rangle$$

$\Rightarrow \quad |\psi_1\rangle = T|\psi\rangle$

$\quad |\psi_2\rangle = |\psi\rangle$

$$\Rightarrow \quad -\langle T\psi | H | \psi \rangle = \langle T\psi | H | \psi \rangle = 0 \;!$$

So TRS copies do not talk! $\Rightarrow$ protection.

<u>If you have inversion</u>

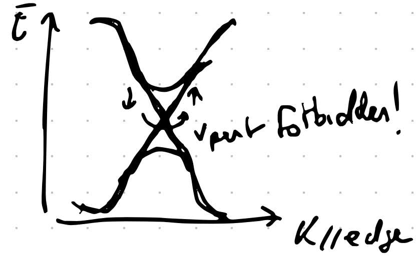
invariant is calculated in same way with inversion

$\nu = \prod_{n\in occ} \cancel{\prod_{k \in TRIMS}} \xi_{kk}^{2n}$ ← pic only one deg band (T and I = deg bands)

$\varepsilon_n^{\uparrow} \xrightarrow{I} \varepsilon_{-n}^{\downarrow} \xrightarrow{T} \varepsilon_n^{\uparrow}$

Edge states

Normally they could back scatter but since $T^2 = -1$ they can't



3D topological insulator

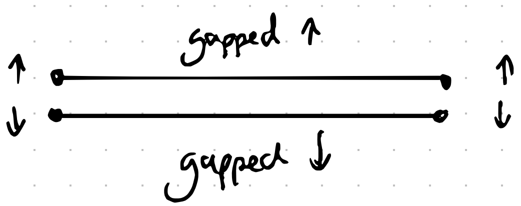
Construct in momentum space:
We start with a lot of symmetries then remove them:

TRS: $T^2 = -1$ $\quad T h_k T^{-1} = h_{-k}$

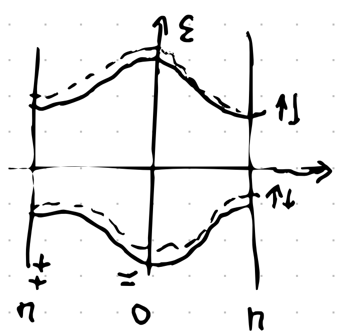
Chiral $U_s h_k U_s^{\dagger} = -h_k$ $\quad S^2 = 1$ (always)

Inversion $I h_k I^{-1} = h_{-k}$

Start with two copies of SSH: spin ↑ spin ↓



End states will be Kramers pairs at $E=0$

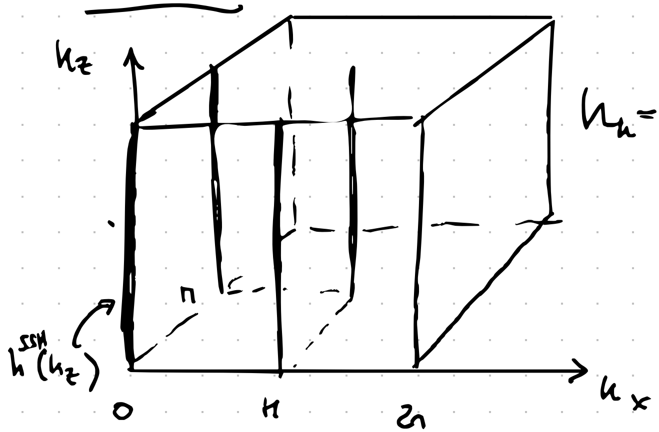


I and TRS = doubly deg bands

$$\varepsilon_{k\uparrow} \xrightarrow{T} \varepsilon_{-k\downarrow} \xrightarrow{I} \varepsilon_{k\downarrow}$$

so $\varepsilon_{k\uparrow} = \varepsilon_{k\downarrow}$

3D BZ



$h_k = h(k_x, K_y, k_z)$

We put a $h_{SSH}(k_z)$ at each $(0,0,k_z)$, $(0,\pi,k_z)$
$(\pi,0,k_z)$, (π,π,k_z)

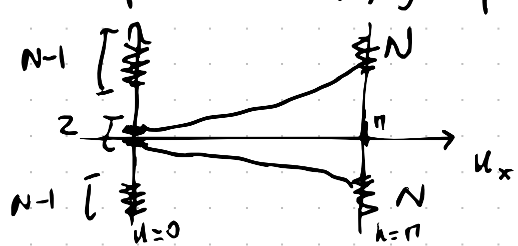
declare that

$$h_k(0,0,k_z) = h^{TOP\ (\delta>0)}_{SSH}(k_z)$$

$$h_k(0,\pi,k_z) = h(\pi,0,k_z) = h(\pi,\pi,k_z) = h^{trivial}_{SSH}(k_z)$$

elsewhere we interpolate between SSH models respecting the symmetries

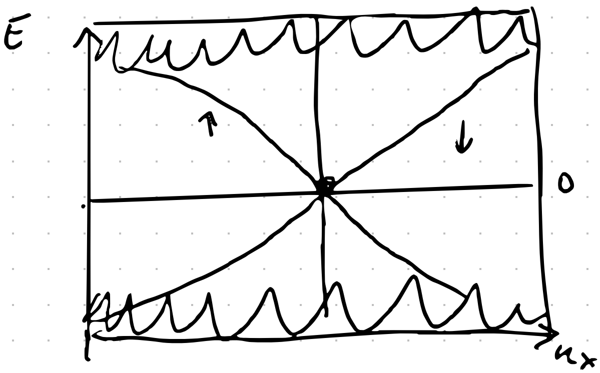
Now p.b.c in x, y open in z: look at one edge



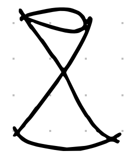
Respects: $\Rightarrow \varepsilon_k = \varepsilon_{-k}$ TRS

$\Rightarrow \varepsilon_k = -\varepsilon_k$ Chiral

Plot the full spectrum



Same for k_y so dispersion @ the surface is a Dirac cone



Two band, gapless $h_{surf} = v_F \vec{k}_{\parallel} \vec{\sigma}$

$T = i\sigma_y$ forces no mass (exercise)

SOL: $i\sigma_y h^*_{surf} (-i\sigma_y) = -v_F k_x \sigma_x - v_F k_y \sigma_y - m\sigma_z \overset{?}{=} h_{-k}$

only if $m = 0$ ($\varepsilon_0 \neq 0$!)

What is protecting the Dirac cone? only TRS (and particle $\not\#$)

- removing chiral symmetry $\Rightarrow$ ε_0 term allowed can shift dispersion up and down
- Invariant? Same! $\nu = \prod_{n \in occ/2} \prod_{k \in TRIM} \xi_n^{2n} = \pm 1$

$\mathbb{Z}_2$ classification.

1D] $\Phi = \sum_n \int \langle u | i\partial_k | u \rangle$

3D] $$\Theta = \sum_{n \in occ} \int \frac{d^3k}{(2\pi)^3} \mathrm{Tr}\left[A_i \partial_j A_k - i\frac{2}{3} A_i A_j A_k \right]$$

$$\vec{A}_{nm} = \langle u_k^n | i \vec{\partial}_k | u_k^m \rangle \equiv$$ non-abelian Berry curvature

$\Theta = 0, \pi$ for $T^2 = -1$

Physical response:

topological magnetoelectric effect

$$F = \epsilon E^2 + \mu^{-1} B^2 + \Theta \vec{E} \cdot \vec{B}$$

(David's lecture?)

AZ classification

$$T h_k T^{-1} = h_{-k} \quad \text{(antiunitary)}$$

$$C h_k C^{-1} = -h_{-k} \quad \text{(antiunitary)}$$

$$S h_k S^{-1} = -h_k \quad \text{(unitary)}$$

$$T^2 = \pm 1 , \quad C^2 = \pm 1 , \quad S^2 = 1 \quad \text{(10 classes)}$$

Class	T	C	S	d=0	d=1	d=2	d=3
A	0	0	0	$\mathbb{Z}$ molecule 1	0	$\mathbb{Z}$ CI	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	1	0	0	$\mathbb{Z}$	0	0	0
BDI	1	1	1	$\mathbb{Z}_2$ molecule 2	$\mathbb{Z}$ SSH	0	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$ kitaev	$\mathbb{Z}$	0
DIII	-1	1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	~~$\mathbb{Z}$~~ $\mathbb{Z}_2$ QSH	$\mathbb{Z}_2$ 3DTI
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
C	0	-1	0	0	0	$\mathbb{Z}$	0
CI	1	-1	1	0	0	0	$\mathbb{Z}$