# Some (Functional) properties of topological matter

David Carpentier
(CNRS and Ecole Normale Supérieure de Lyon)

# Scope

- **Functional properties**
  - I asked ChatGPT
  - I decided to focus on
    - Topological insulators and semi-metals
    - ~~topological superconductors~~

Andrej Mesaros, Tristan Cren

Alexandre Pourret

Paul McClarty, Laura Messio, Stanislas Rohart

Hi buddy. Can you tell me what's a functional property of a material ?

A functional property of a material refers to a specific characteristic that defines its performance in particular applications. These properties are often related to how a material responds under certain conditions or how it interacts with its environment. They are critical in determining the suitability of a material for a specific purpose.

Functional properties can include:

1. **Electrical Properties**: Characteristics like conductivity, resistivity, and dielectric constant that determine how a material behaves in an electric field.
2. **Thermal Properties**: Characteristics such as thermal conductivity, thermal expansion, and heat capacity that determine how a material conducts, absorbs, and retains heat.

4. **Optical Properties**: Characteristics such as refractive index, transparency, and reflectivity that determine how a material interacts with light.
5. **Magnetic Properties**: Characteristics such as magnetic permeability and coercivity that define how a material responds to a magnetic field.

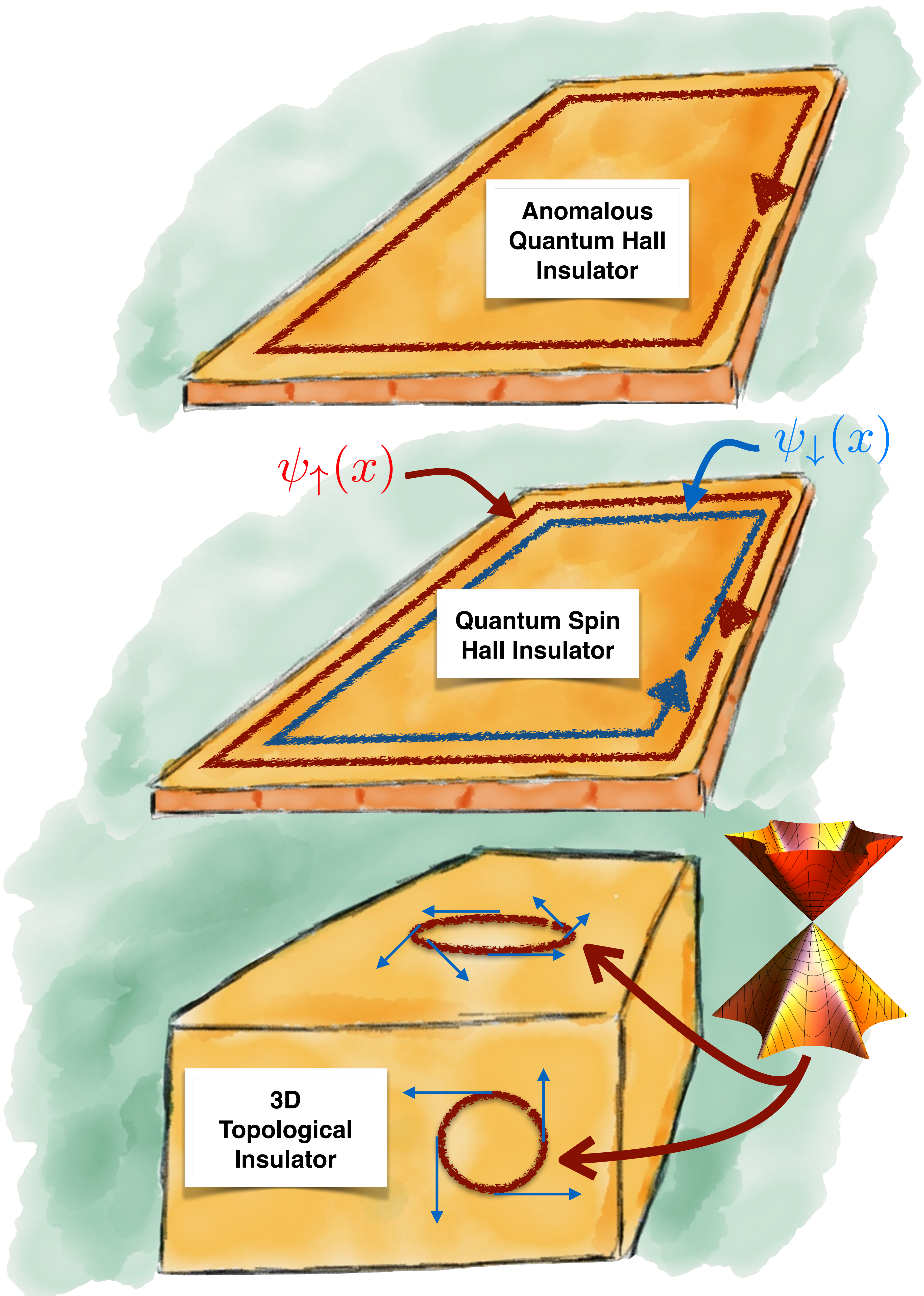
# Scope and Outline

- **Functional properties**
  - I asked ChatGPT
  - I decided to focus on
    - Topological insulators and semi-metals
    - ~~topological superconductors~~

- **2D topological insulators**
  - Edge states versus bulk: the Quantum Hall Effect example
  - Modified electrodynamics and quantized Hall response in 2D
- **3D topological insulators**
  - Surface properties
  - Modified electrodynamics in 3D: magneto-electric effects
  - (A choice of some) consequences
- **3D topological semimetals**
  - Recap of topological properties: anomalous Hall effect
  - Modified electrodynamics
  - Non-reciprocity of optical and thermal properties

# Topological Boundary States

- Quantum Hall Insulator
  - Two dimensions
  - breaks Time-Reversal Symmetry
  - Chern index
  - ➡ **Chiral edge states**
- Quantum Spin Hall Insulator
  - Two dimensions
  - Time-Reversal Symmetry + spins 1/2
  - Kane-Mele $Z_2$ index
  - ➡ **Helical edge states : Kramers pair**
- 3D Topological Insulators
  - Three dimensions
  - Time-Reversal Symmetry + spins 1/2
  - Kane-Mele $Z_2$ index
  - ➡ (odd number of) **Dirac cone**



# Robustness of edge States

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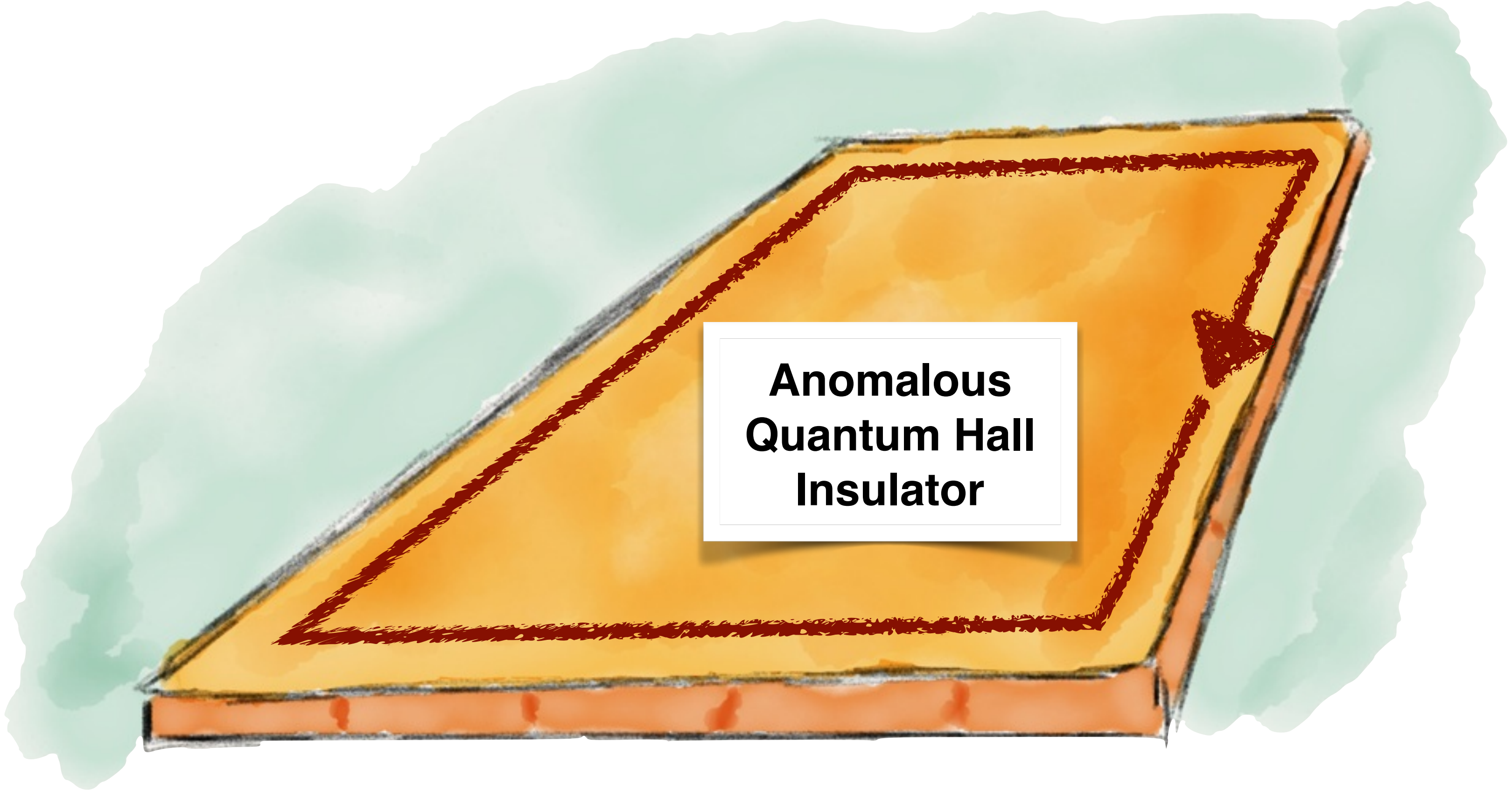
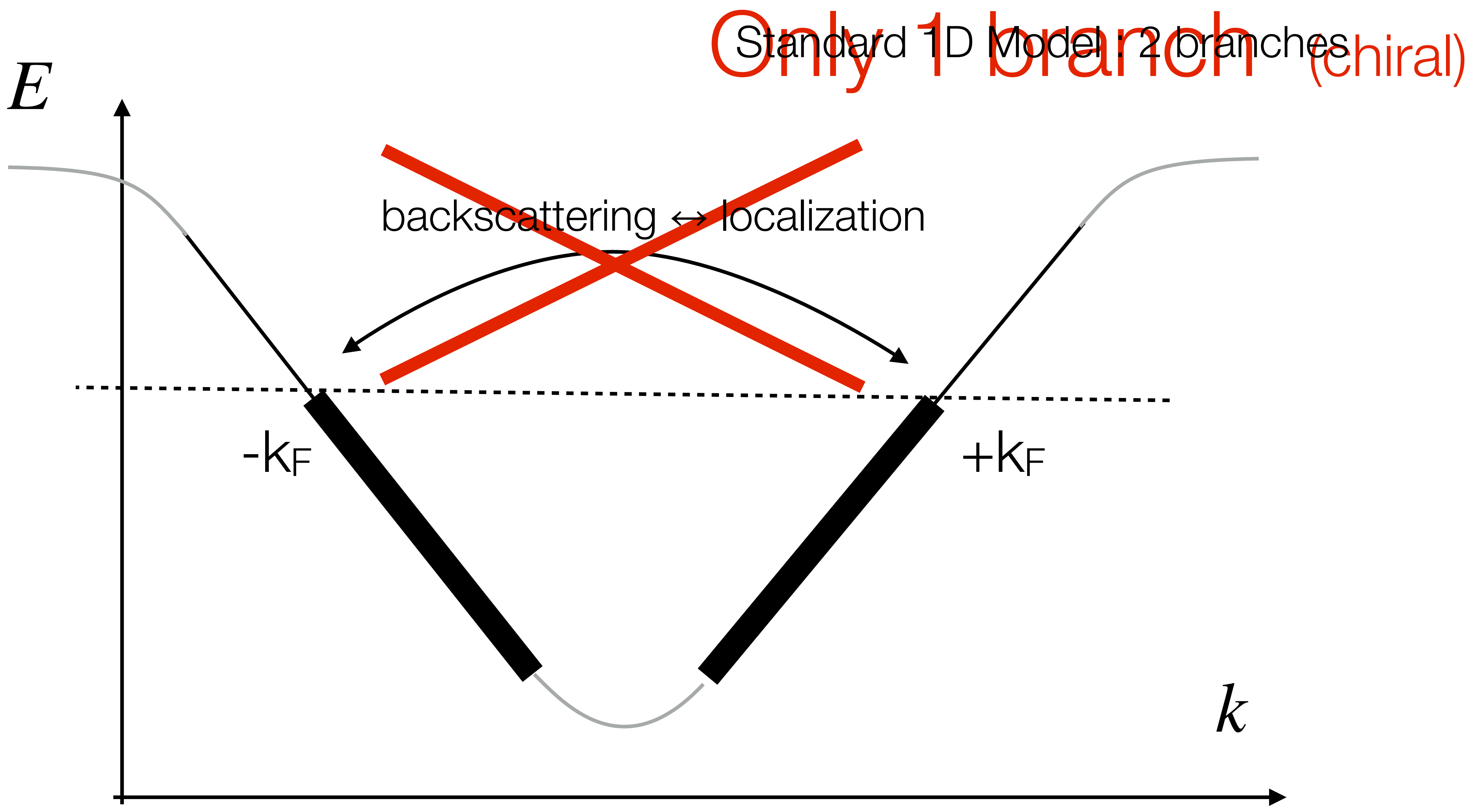
  - ➡ **Chiral edge states**




<u>Robustness of edge states :</u>

no backscattering because chiral modes

> Robustness: chirality of the modes (T-breaking)

# Robustness of edge States

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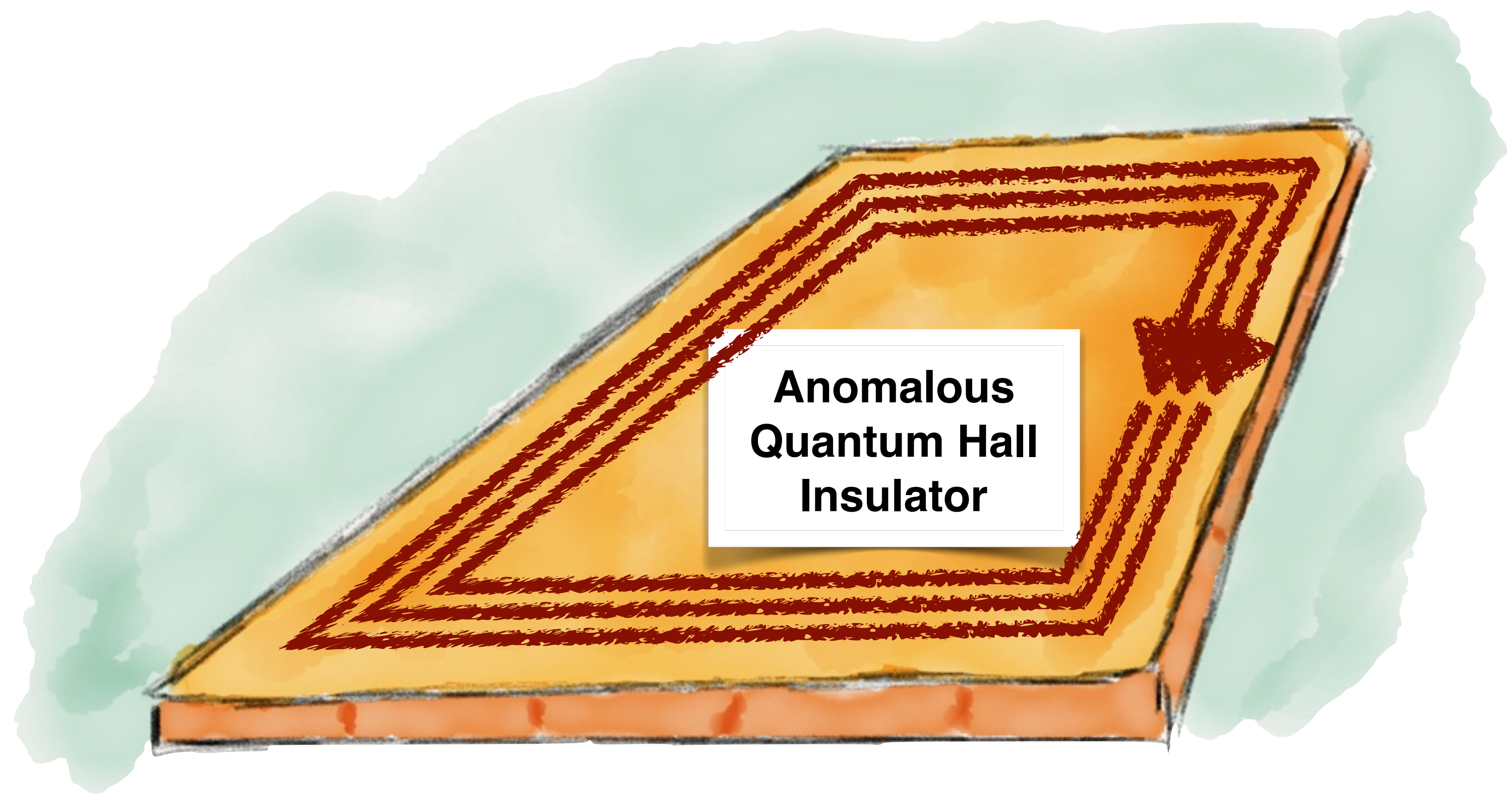


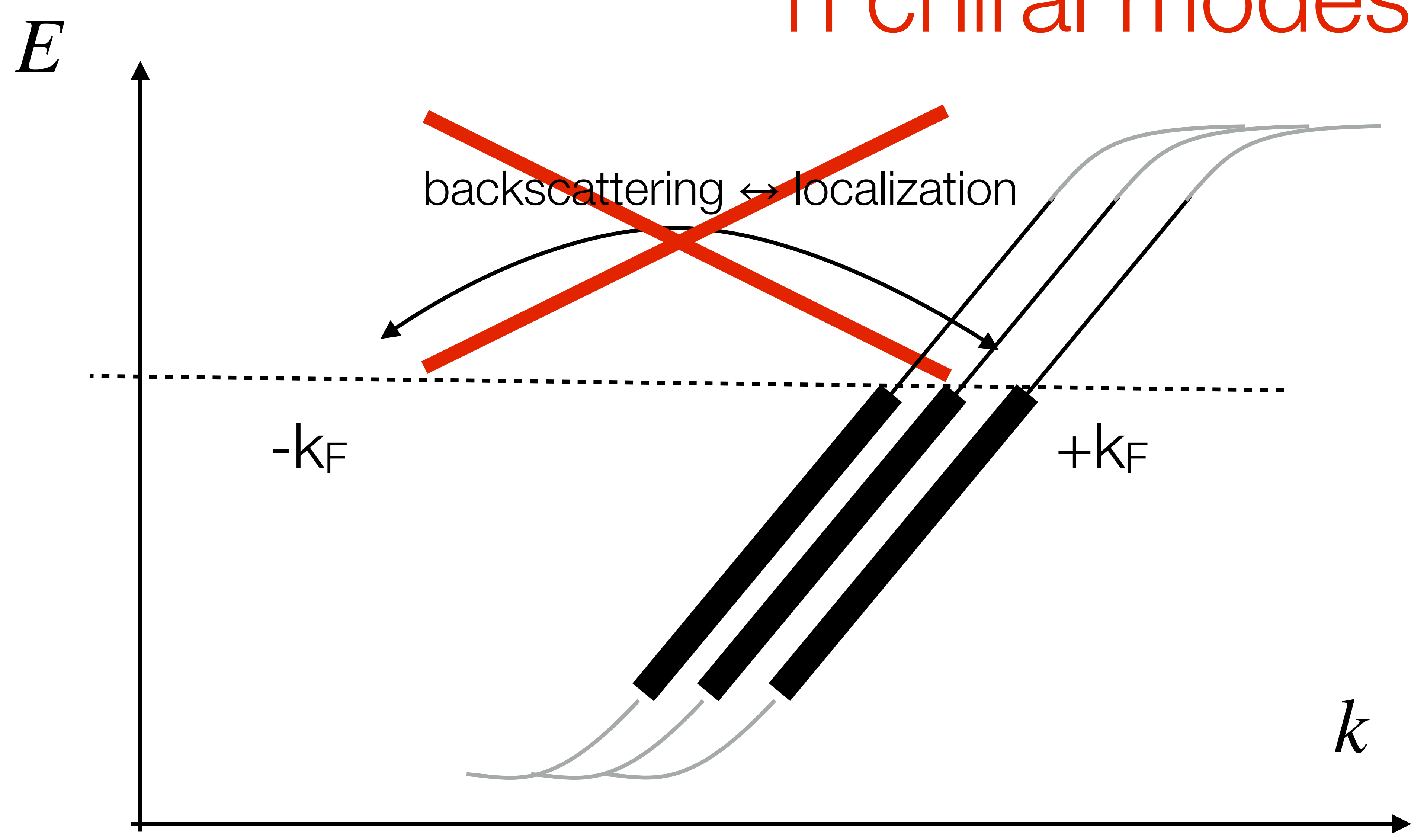

Robustness of edge states :

no backscattering because chiral modes

- Top. Index $n \in \mathbb{Z}$
- all modes are ballistic

Robustness: chirality of the modes (T-breaking)

n chiral modes



# Topological Boundary States

C.L.Kane and E.J.Mele, PRL 95, 226801 (2005)

- Quantum Spin Hall Insulator
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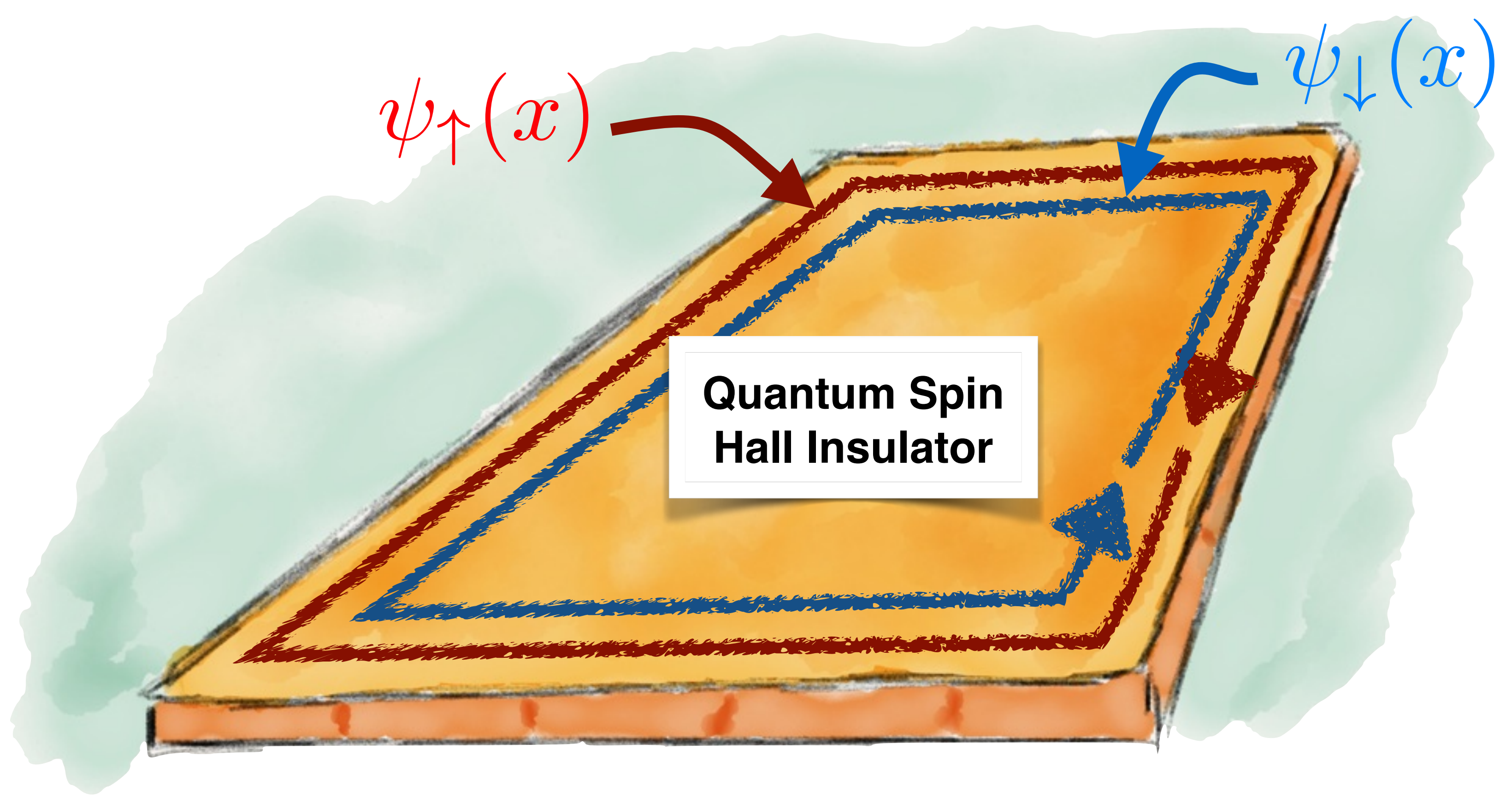
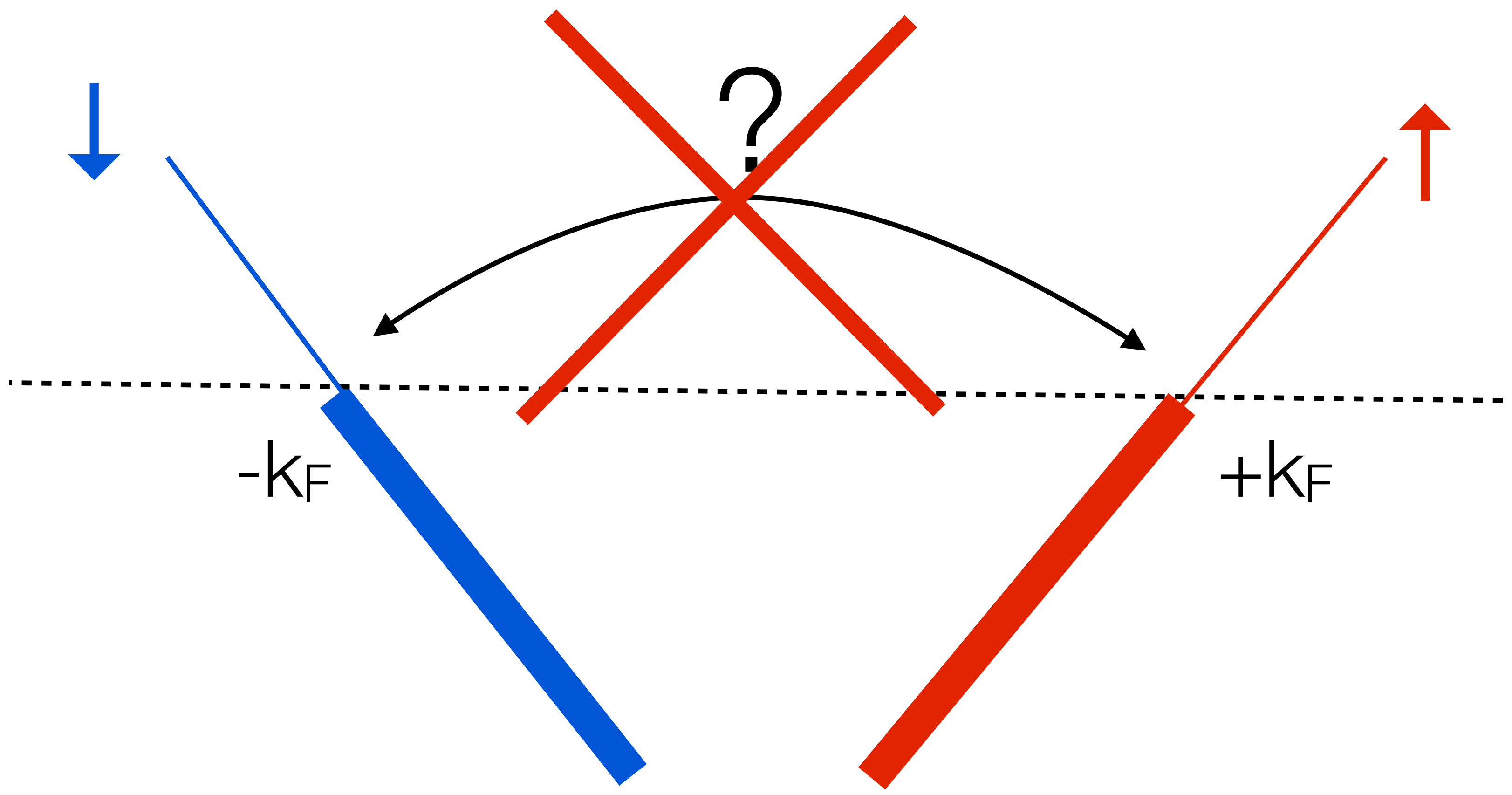



«Protected Edge States» : robust properties, remain ballistic




2 branches, but ≠ spins : no backscattering
(protected by T reversal symmetry)
⇒ topological robustness

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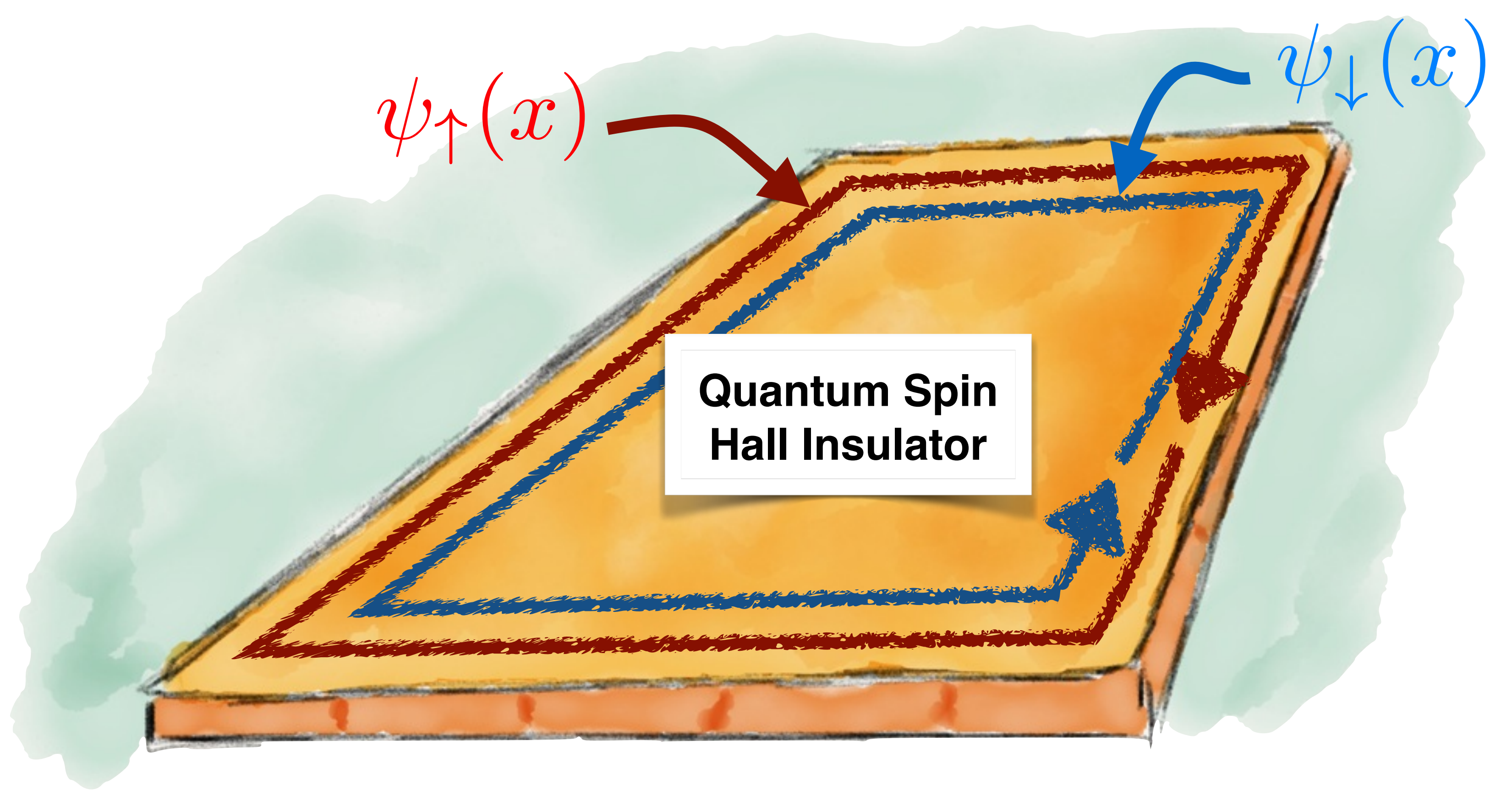
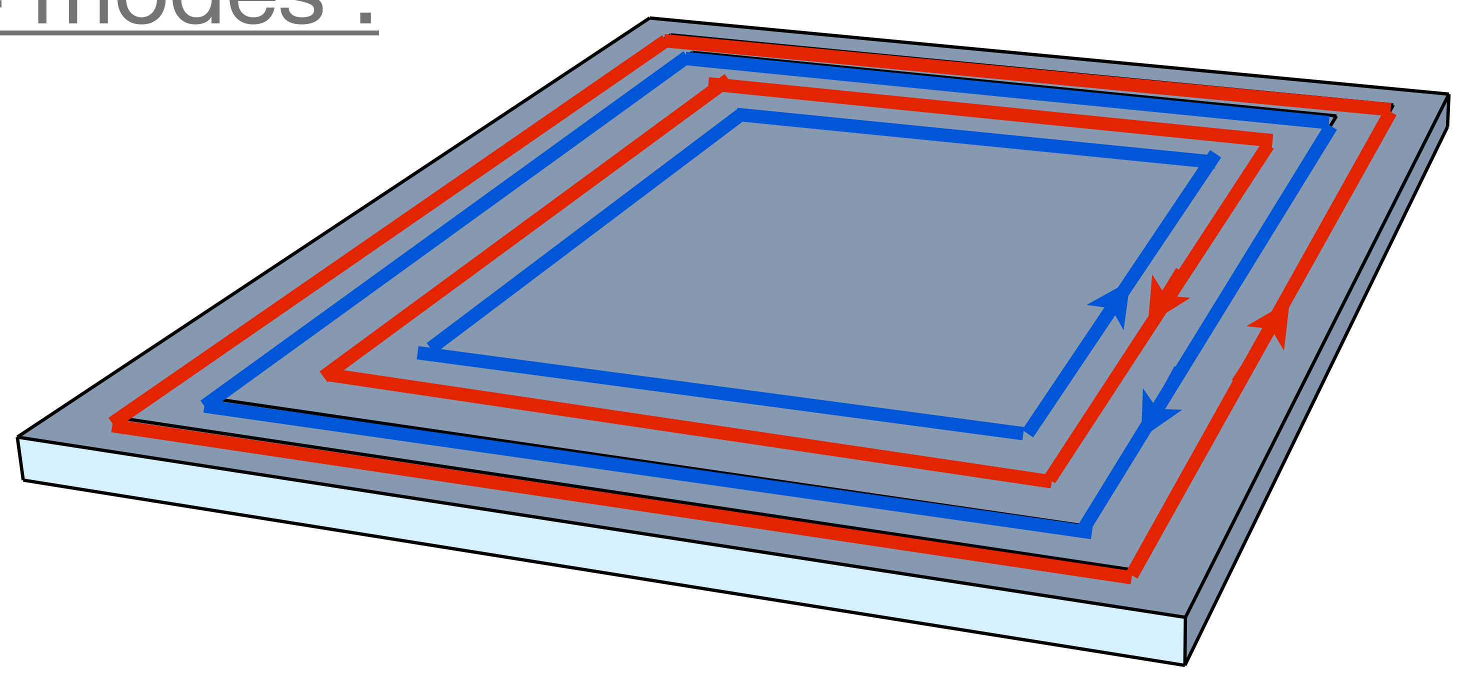
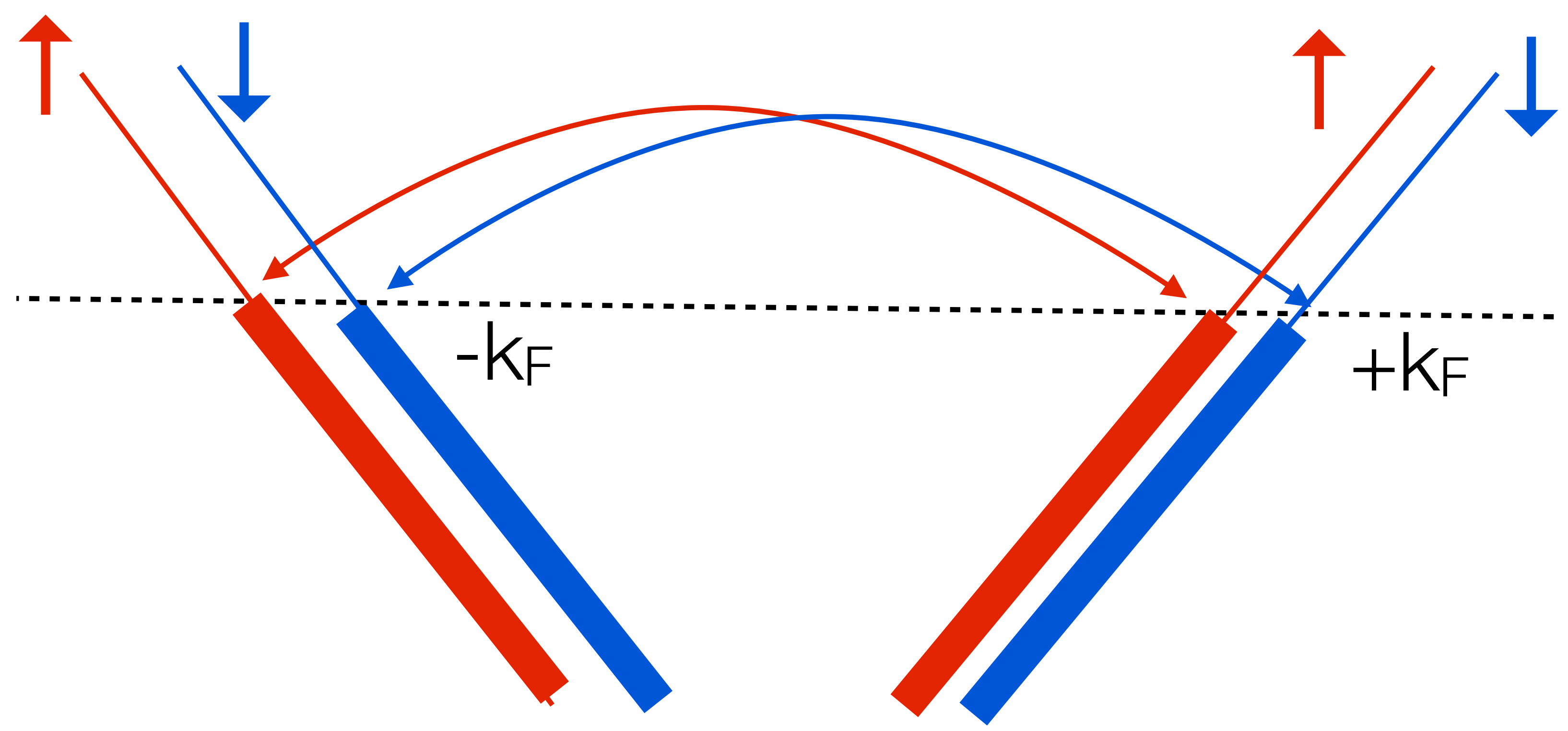



with 4 modes :

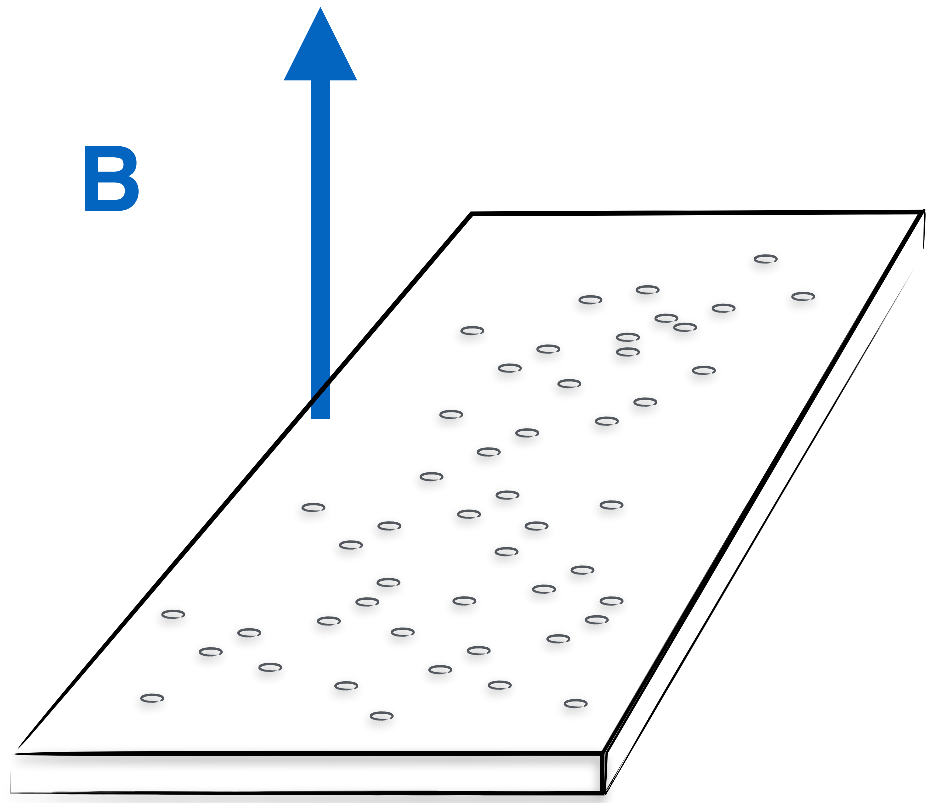





Backscattering allowed ⇒ No topological order

$Z_2$ Topological index for QSHE : yes / no

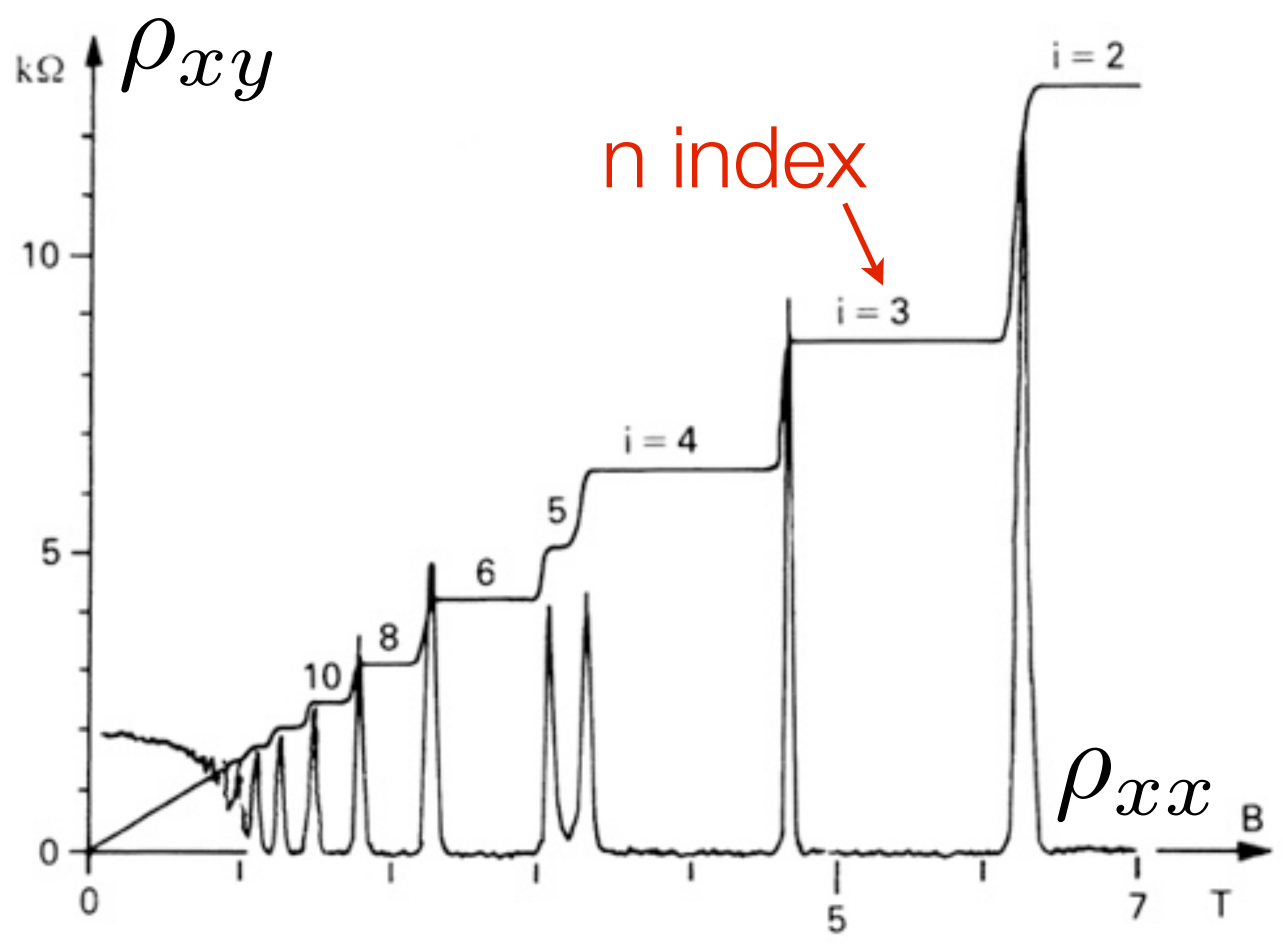
# Quantum Hall Effect and Chern Topological Insulator




2DEG (Heterojunction GaAs/AlGaAs)

$$\sigma_{xy} = n\frac{e^2}{h}$$

with high precision ($10^{-9}$)



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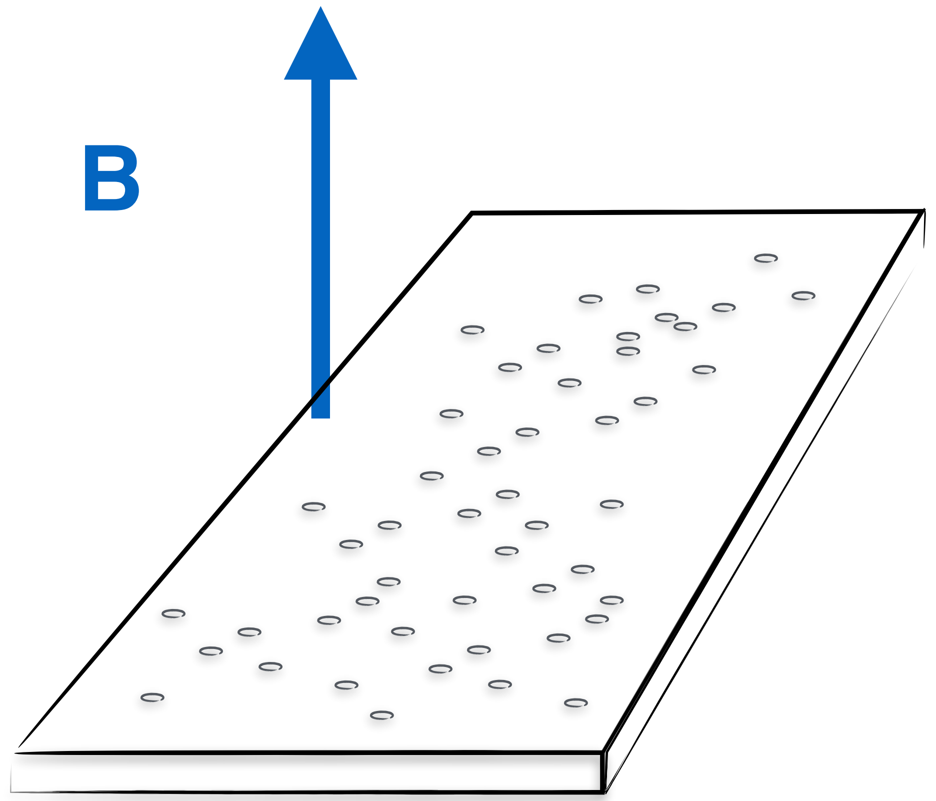


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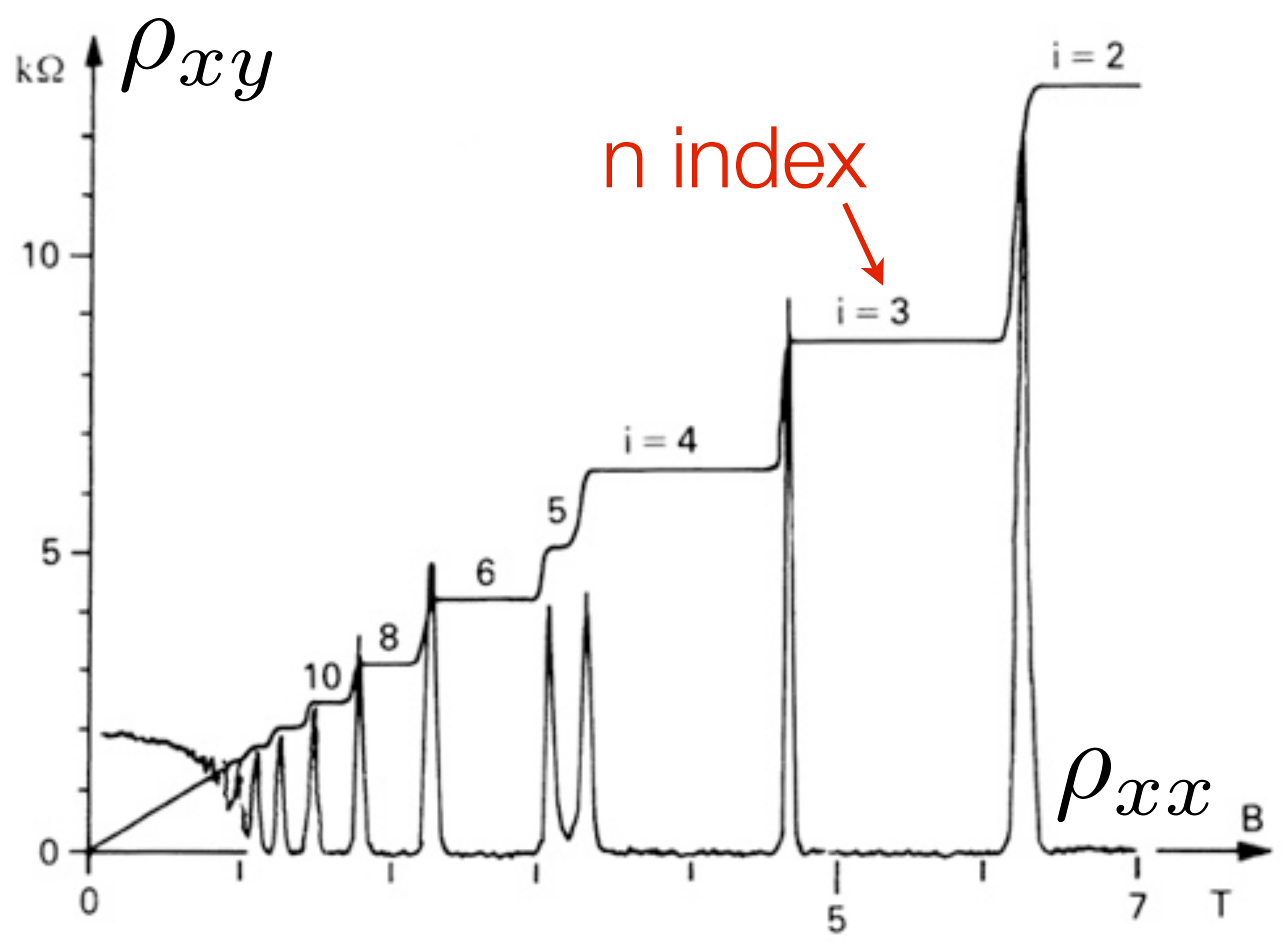


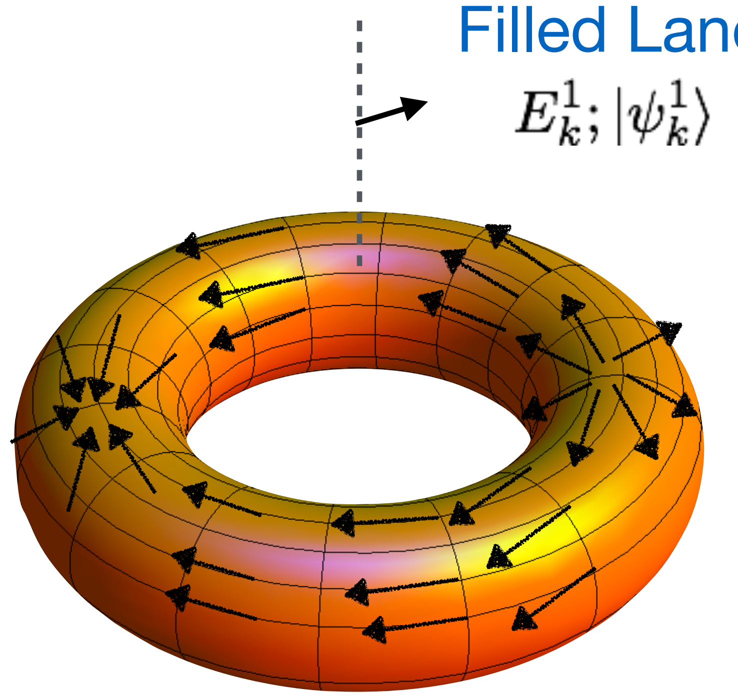

n is a topological invariant

Thouless *et al.*. PRL **49** (1982)

## Topology of Vector Bundle over the Brillouin/Boundary conditions Torus

- Chern number n=$C_1 = \dfrac{1}{2\pi}\displaystyle\int dS\ F$

  with $F$ the Berry curvature



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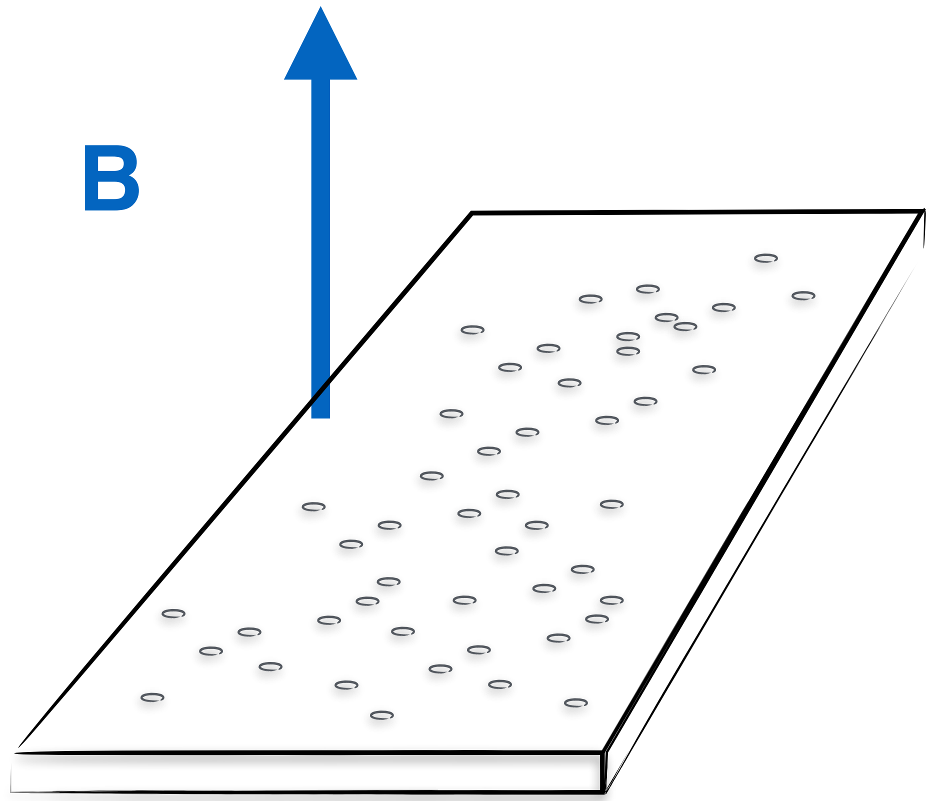


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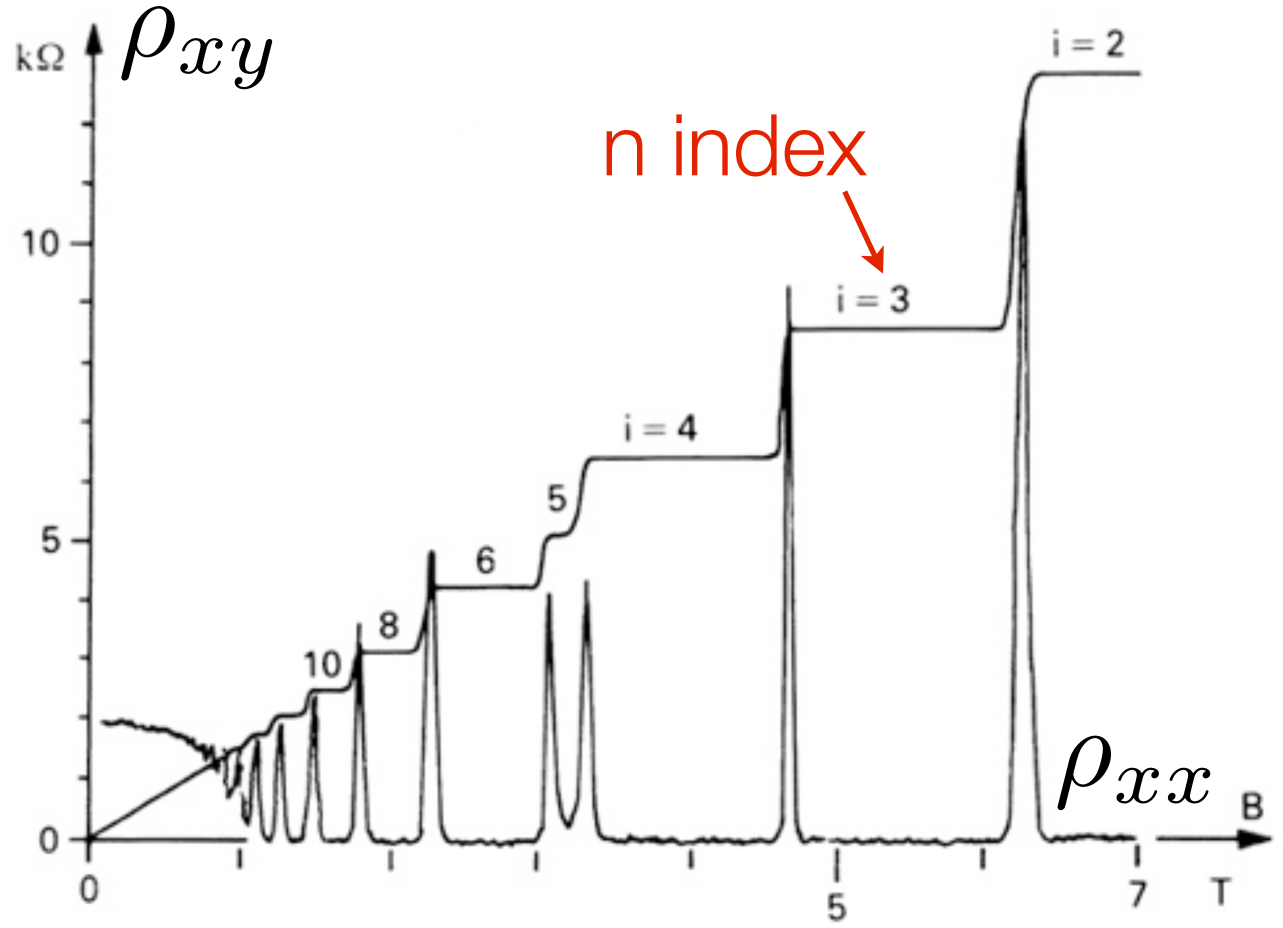


n is a topological invariant ↔ number of (robust) edge modes

Thouless *et al.*. PRL **49** (1982)

M. Büttiker, PRB **38** (1988)

Topology of Vector Bundle
over the Brillouin/Boundary conditions Torus

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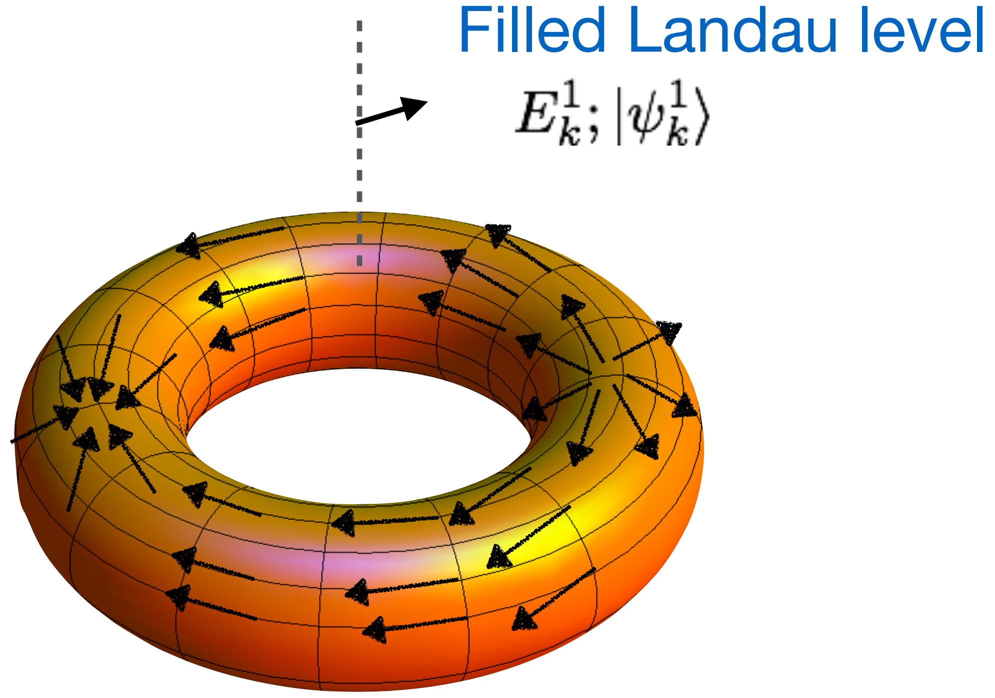
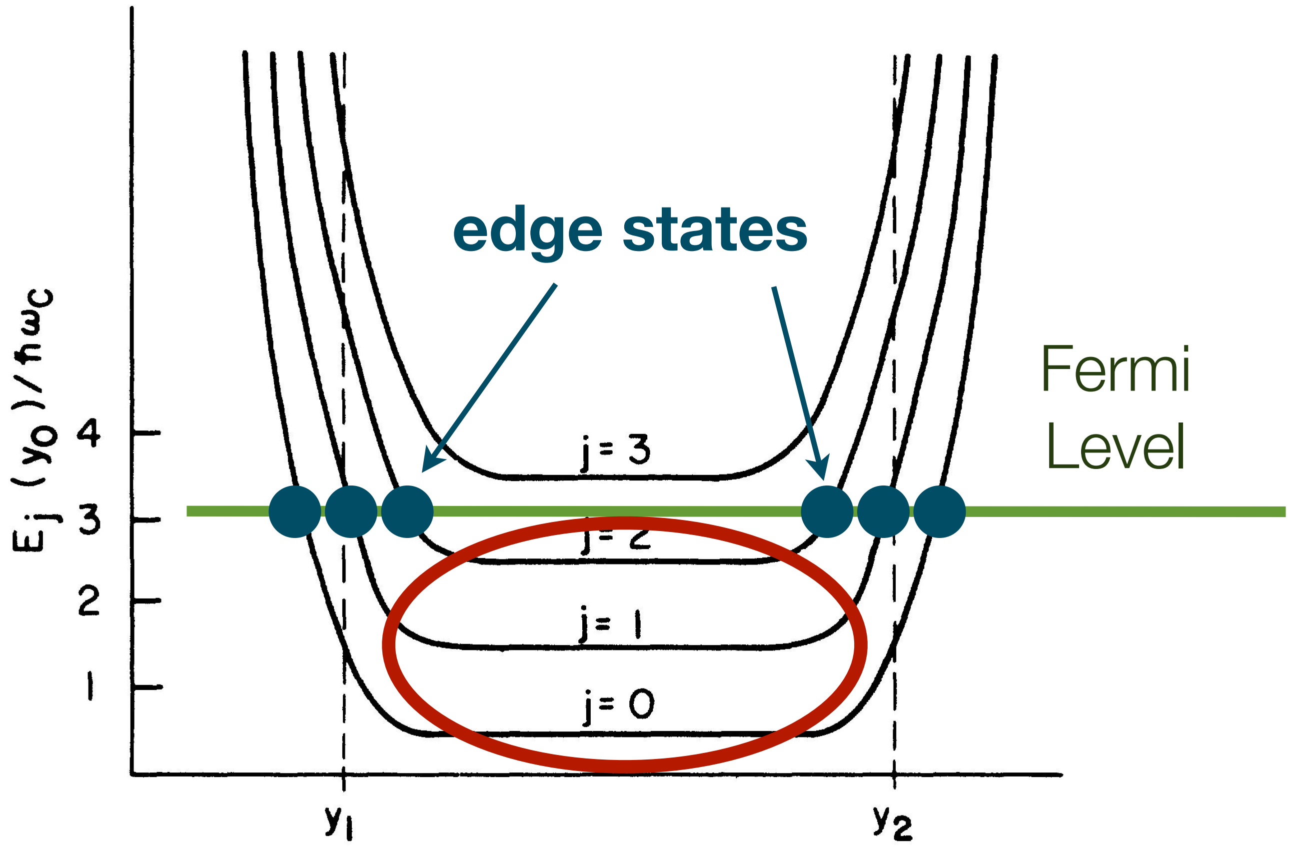
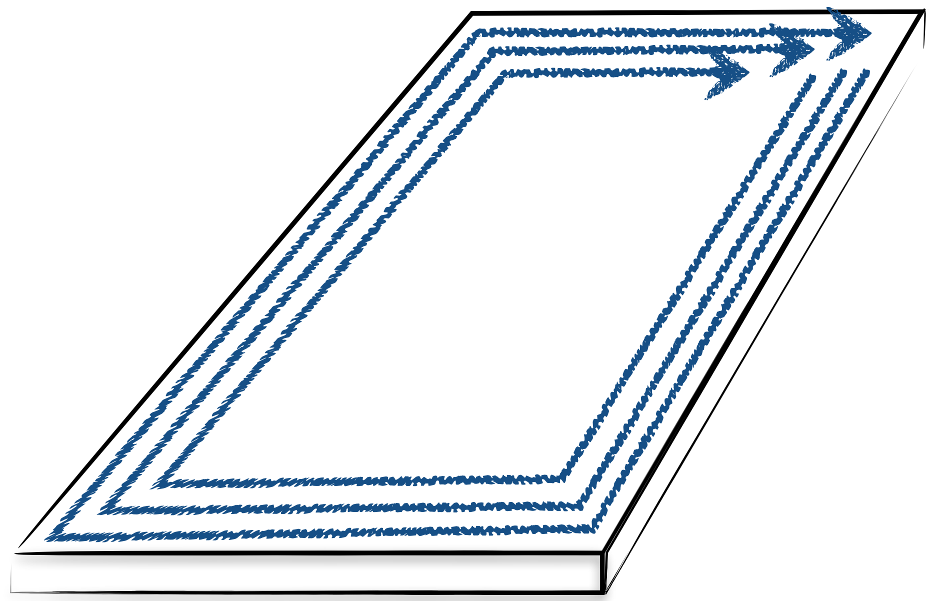


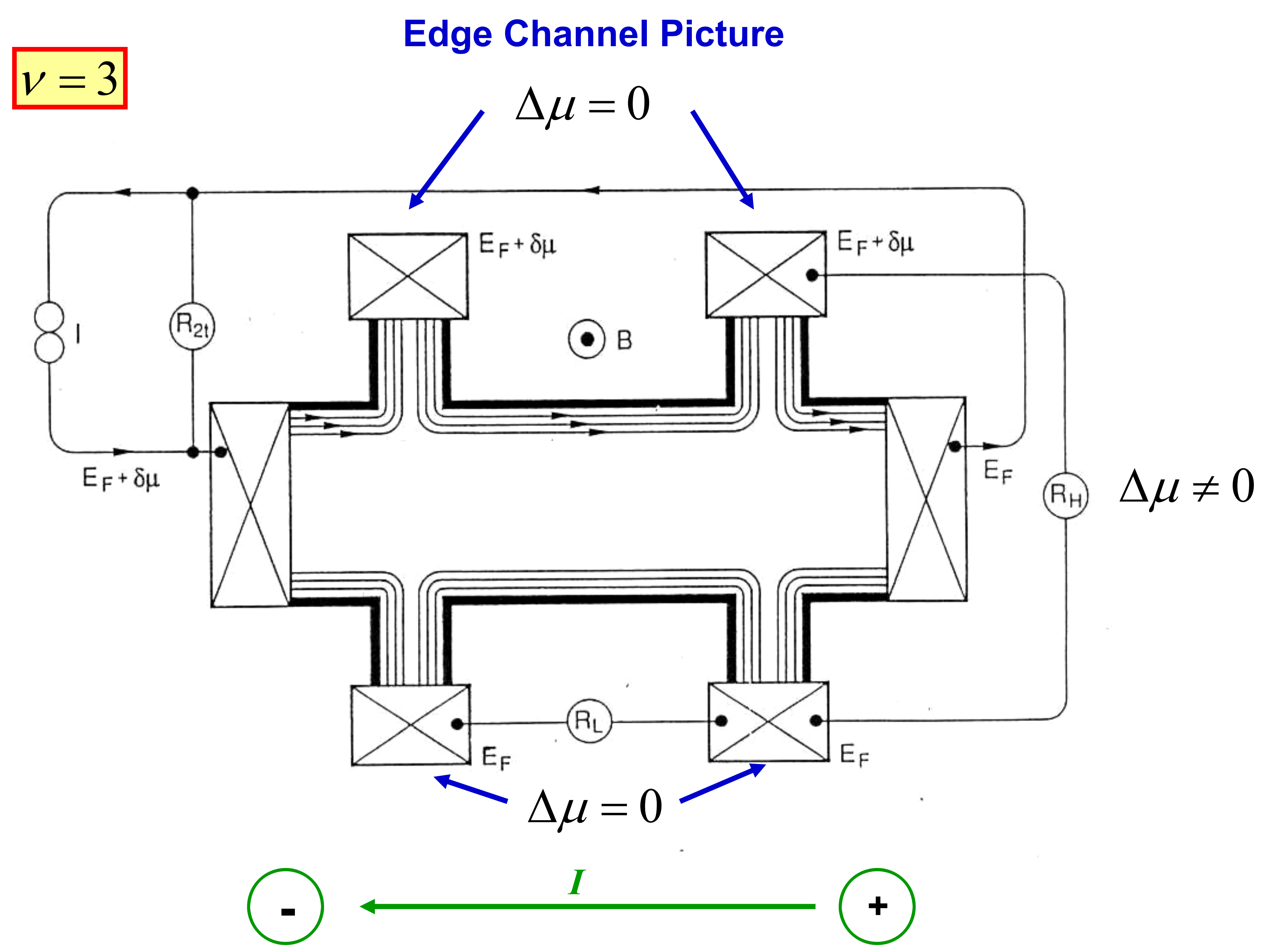




Filled Landau Levels



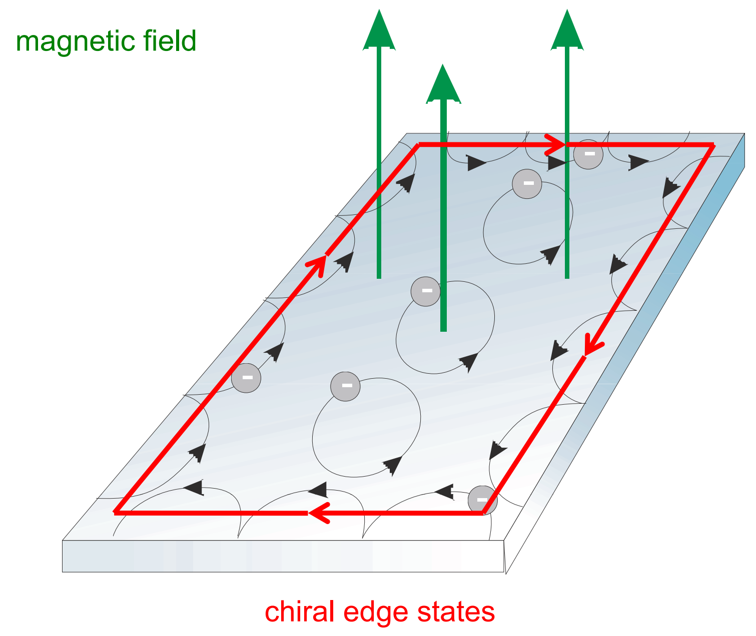
# Transport and edge states

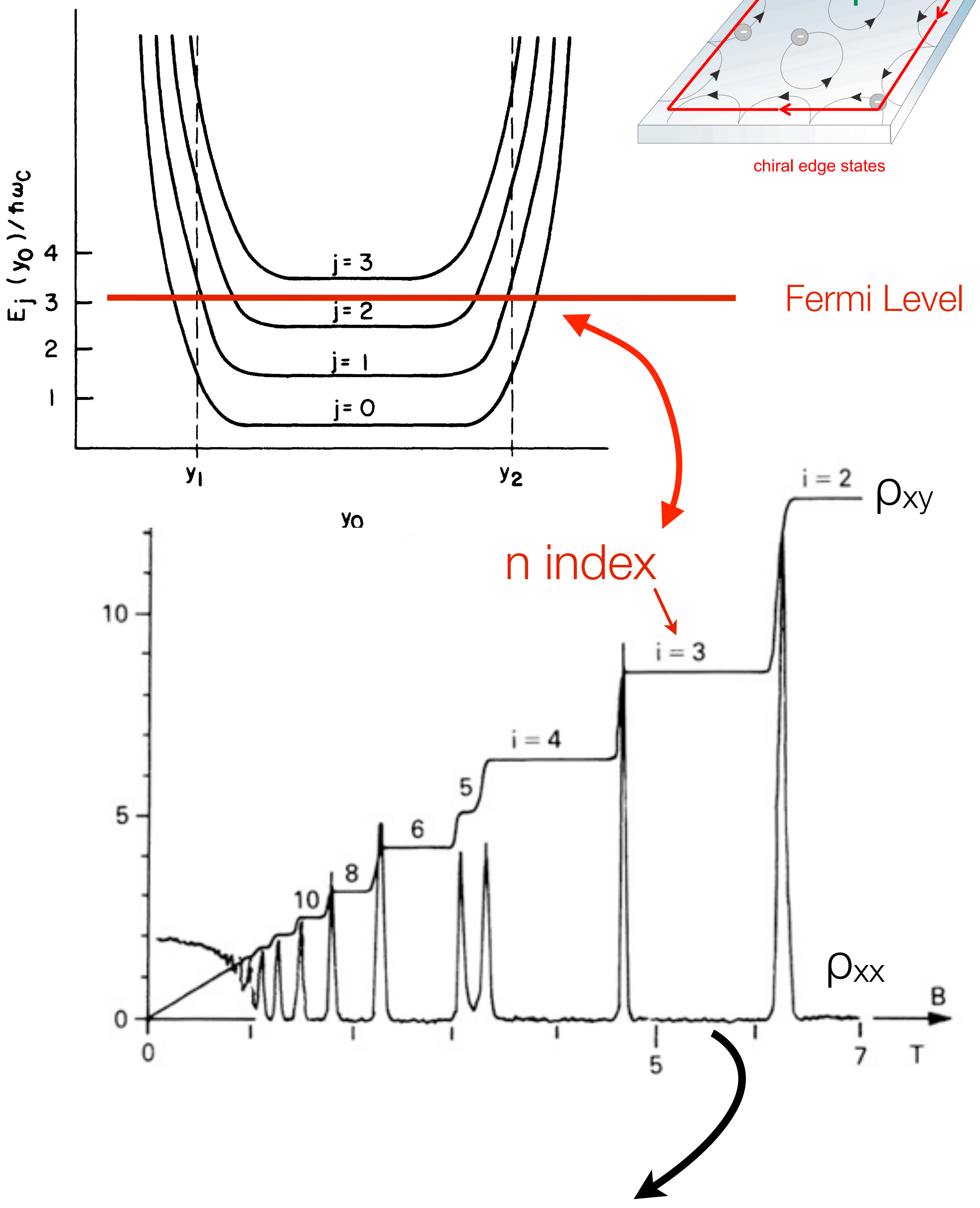
- **Landauer-Buttiker formalism**
  - Multi-terminal geometry




Chirality + n →conductance matrix → multi-terminal conductances

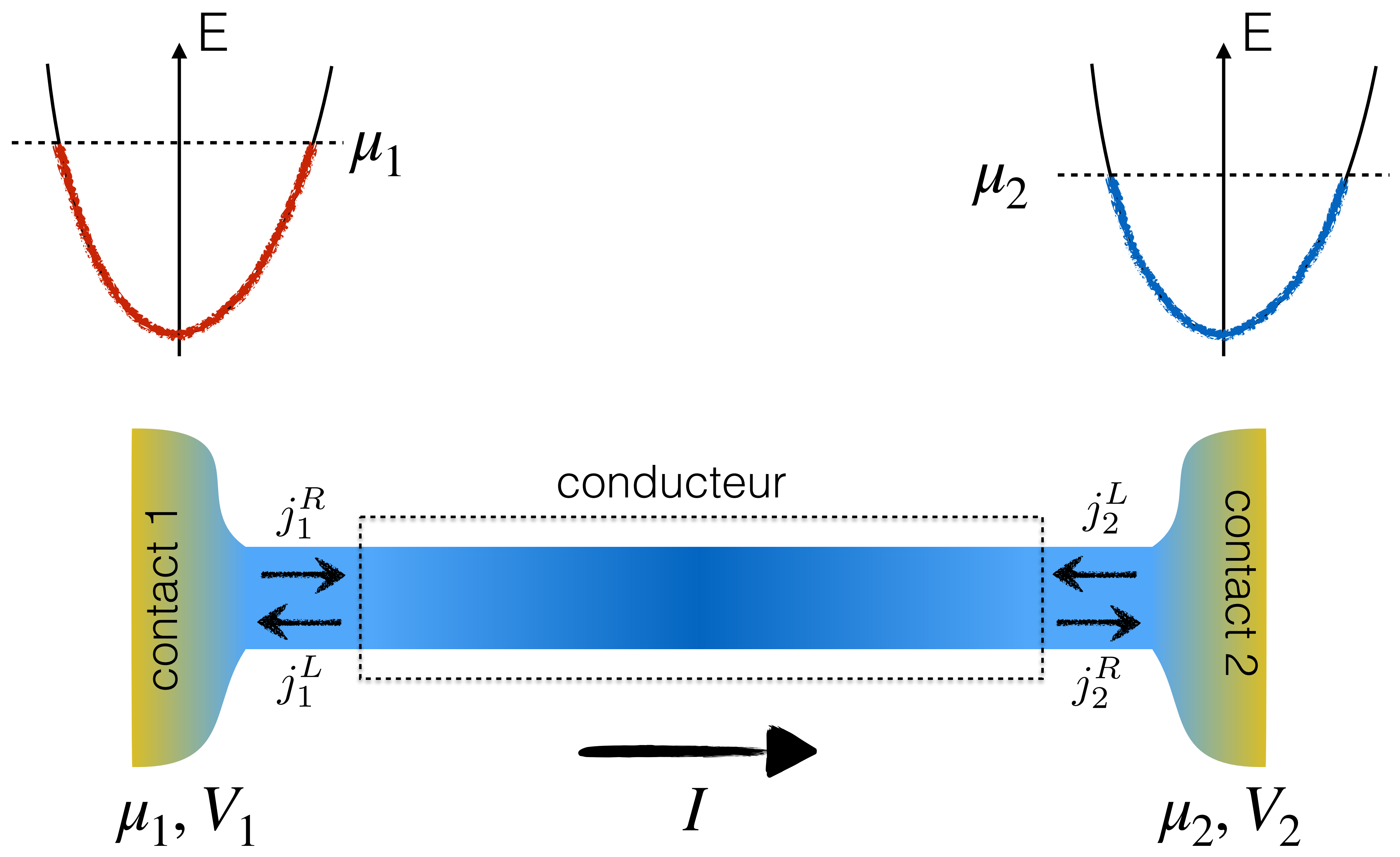
M. Büttiker, PRB **38** (1988)




No longitudinal resistivity : Insulator

# Transport and edge states




## Landauer-Buttiker formalism

- 2-terminal geometry in D=1
- $\begin{pmatrix} j_1^L \\ j_2^R \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} j_1^R \\ j_2^L \end{pmatrix}$ with $R + T = 1$
- Total current is $I = j_2^R - j_2^L = Tj_1^R + (R-1)j_2^L = T(j_1^R - j_2^L)$

  Outgoing    Incoming

- Incoming current (from lead): $j_1^R \left( = (-e)n_1(\mu_1)v_1 \right) = (-e)\int \frac{dk}{(2\pi)} \underbrace{\frac{1}{\hbar}\frac{\partial E_\mathbf{k}}{\partial k}}_{\text{Velocity } v_k} f_{\mu_1}(E_\mathbf{k}) = \frac{(-e)}{h}\int dE\, f_{\mu_1}(E)$
- Total current is $I = T\frac{(-e)}{h}\int dE \left[ f_{\mu_1}(E) - f_{\mu_2}(E) \right] = T\frac{(-e)}{h}(\mu_1 - \mu_2) = T\frac{e^2}{h}(V_1 - V_2)$
- Conductance: $G = \frac{I}{\Delta V} = T\frac{e^2}{h} = T\, G_0,$

$G_0 = e^2/h = 3.87\ 10^{-5}\ S$

Von Klitzing constant $R_K = h/e^2 = 25\ 812.807\ \Omega$

# Transport and edge states

- **Landauer-Buttiker formalism**
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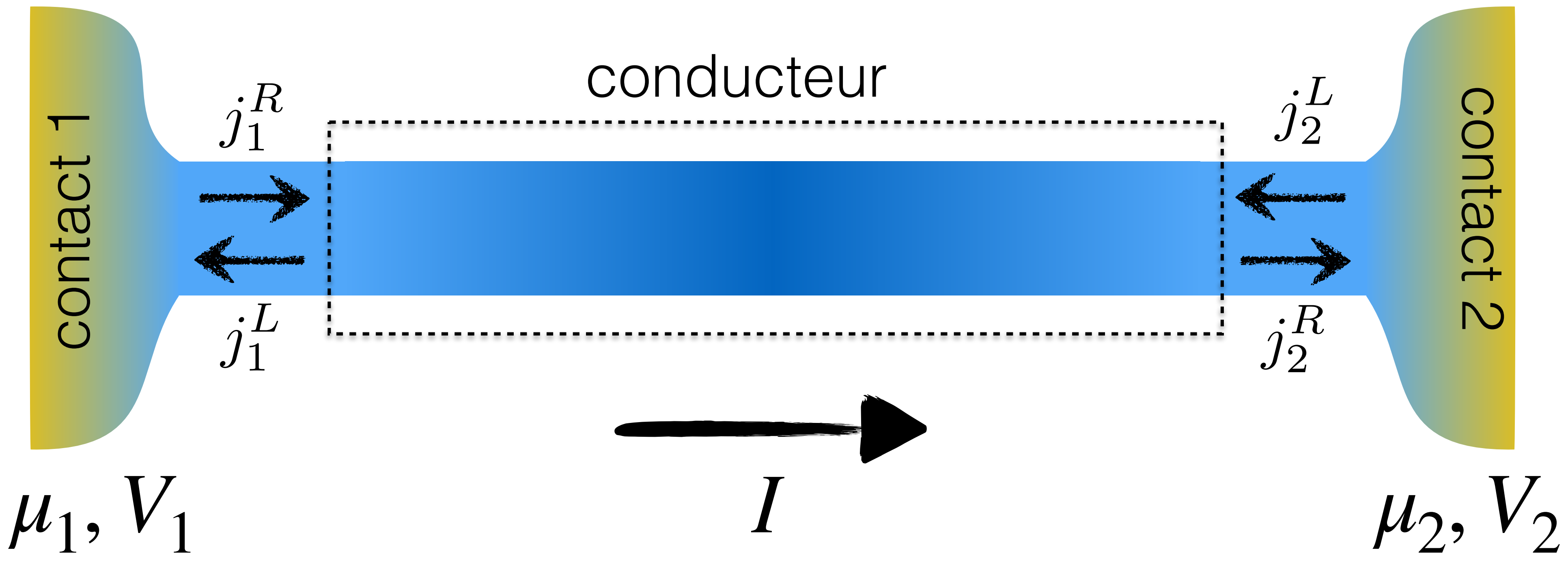
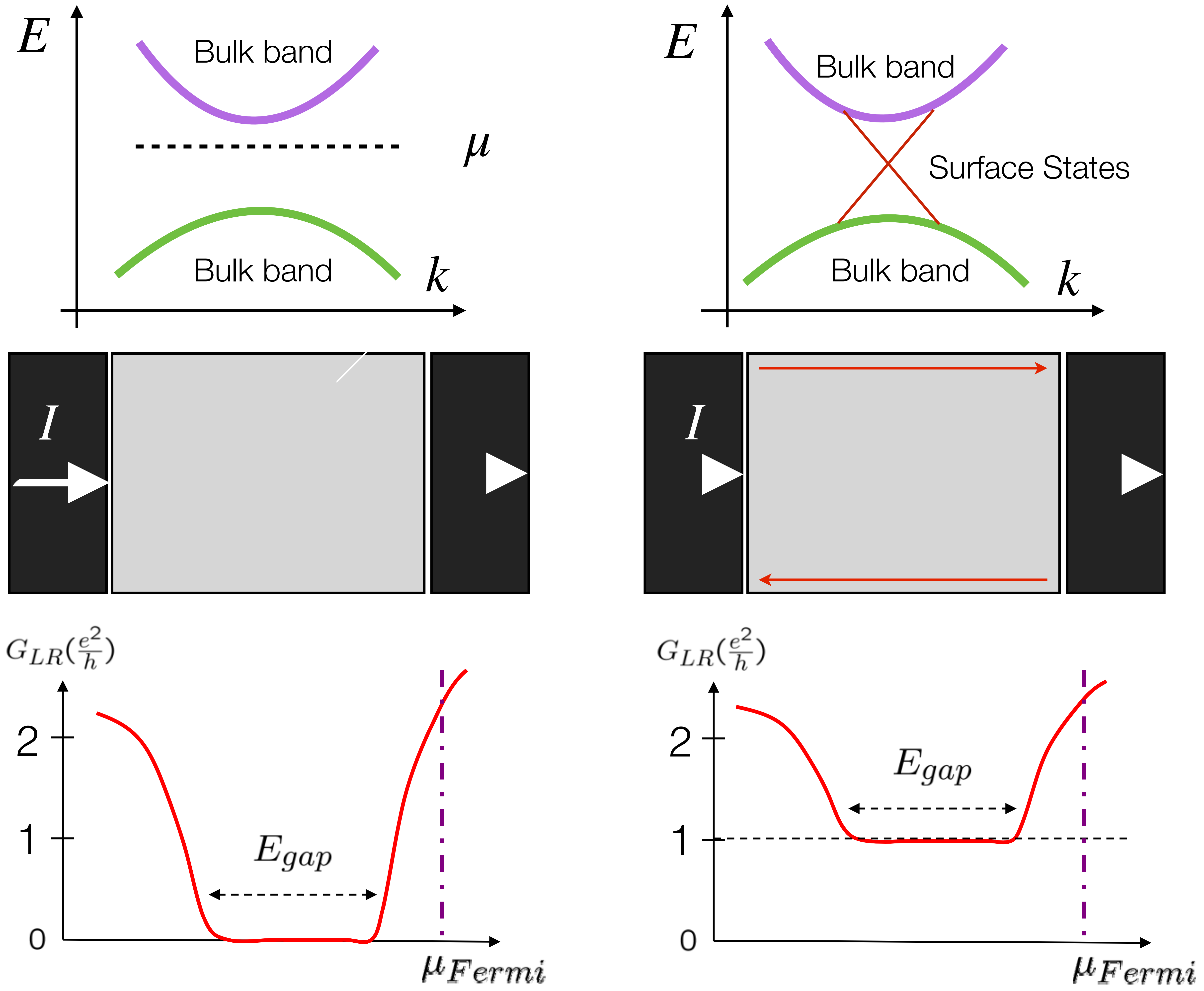
  - Conductance: $G = \dfrac{I}{\Delta V} = T\dfrac{e^2}{h} = T\ G_0,$
- **Chern insulator (Quantum Hall effect)**
  - Effective transmission through the edges
  - When $\mu \in$ gap, $T = n$ (number of edge states)

$G_0 = e^2/h = 3.87\ 10^{-5}\ S$

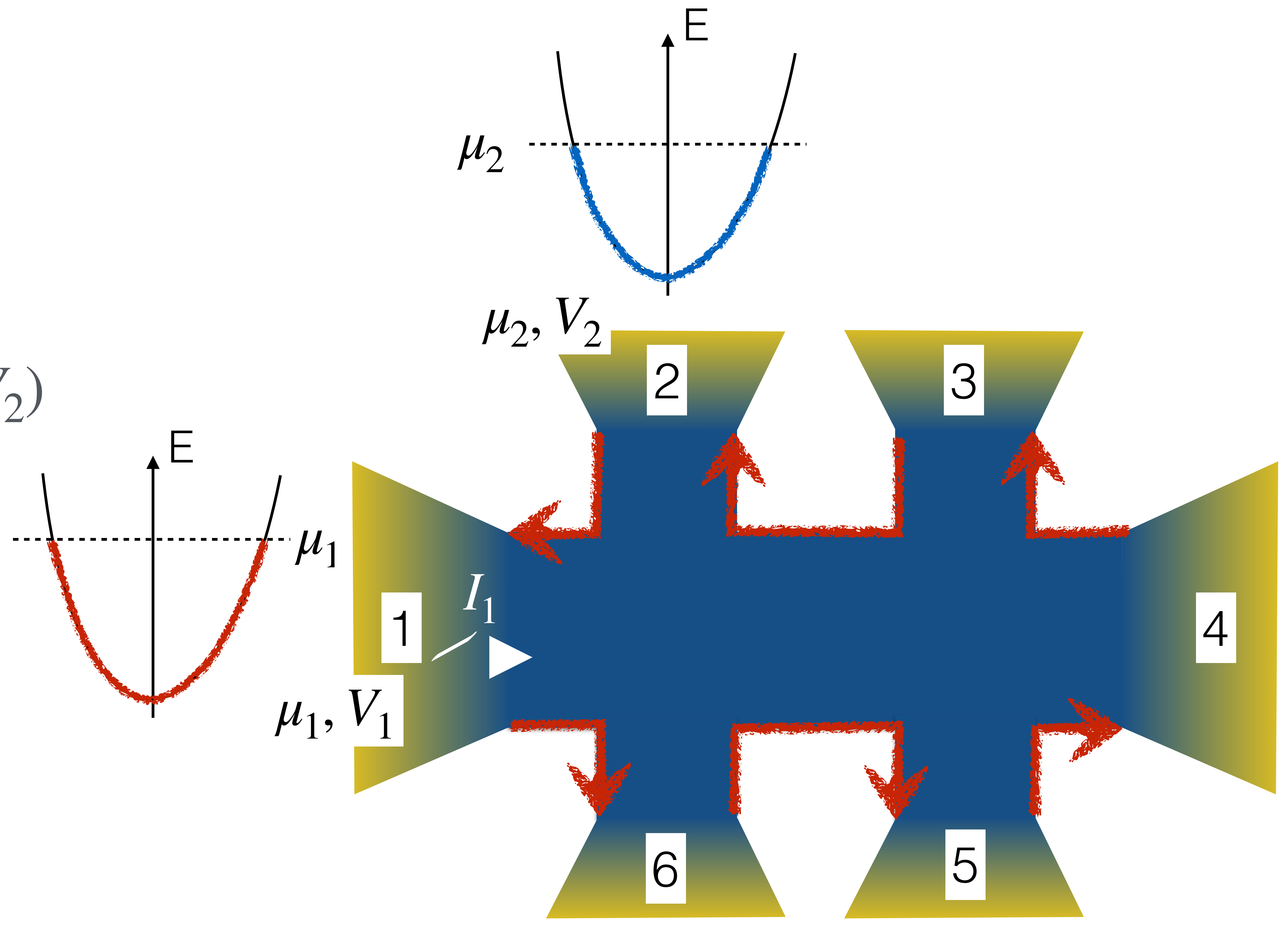
Von Klitzing constant $R_K = h/e^2 = 25\ 812.807\ \Omega$



# Transport and edge states

- **Landauer-Buttiker formalism**
  - 2 terminal: $I = T\frac{(-e)}{h}(\mu_1 - \mu_2) = T\frac{e^2}{h}(V_1 - V_2)$
  - Multi-contacts: $I_\alpha = \frac{-e}{h}\sum_{\beta\neq\alpha} T_{\alpha\beta}\left(\mu_\beta - \mu_\alpha\right)$




- **Quantum Hall effect**
  - $n$ chiral edges modes (linear response):

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} = \frac{(-e)n}{h}\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix}$$

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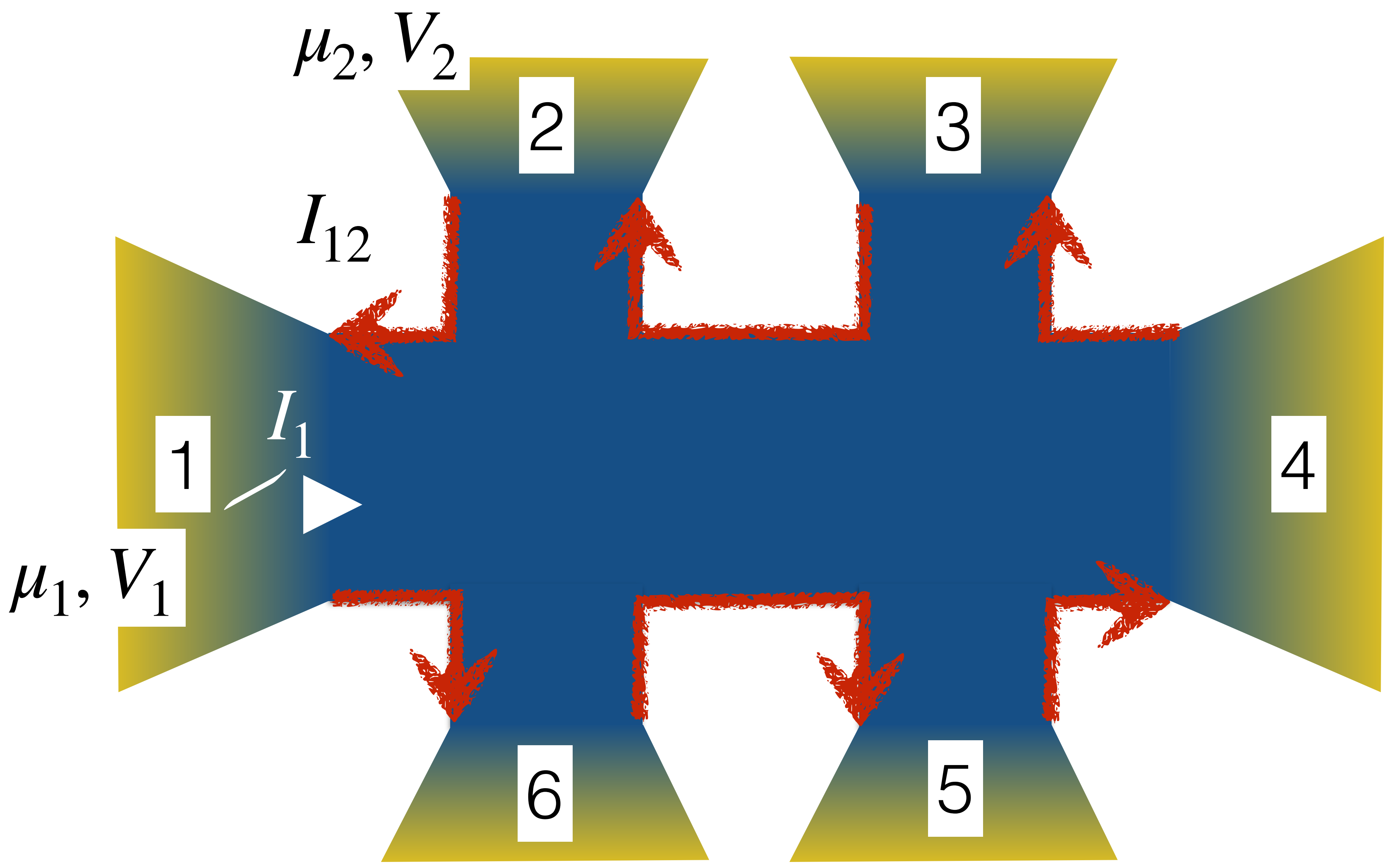



  - Current conservation: $\sum_\alpha I_\alpha = 0$ and definition of $\mu = 0$:

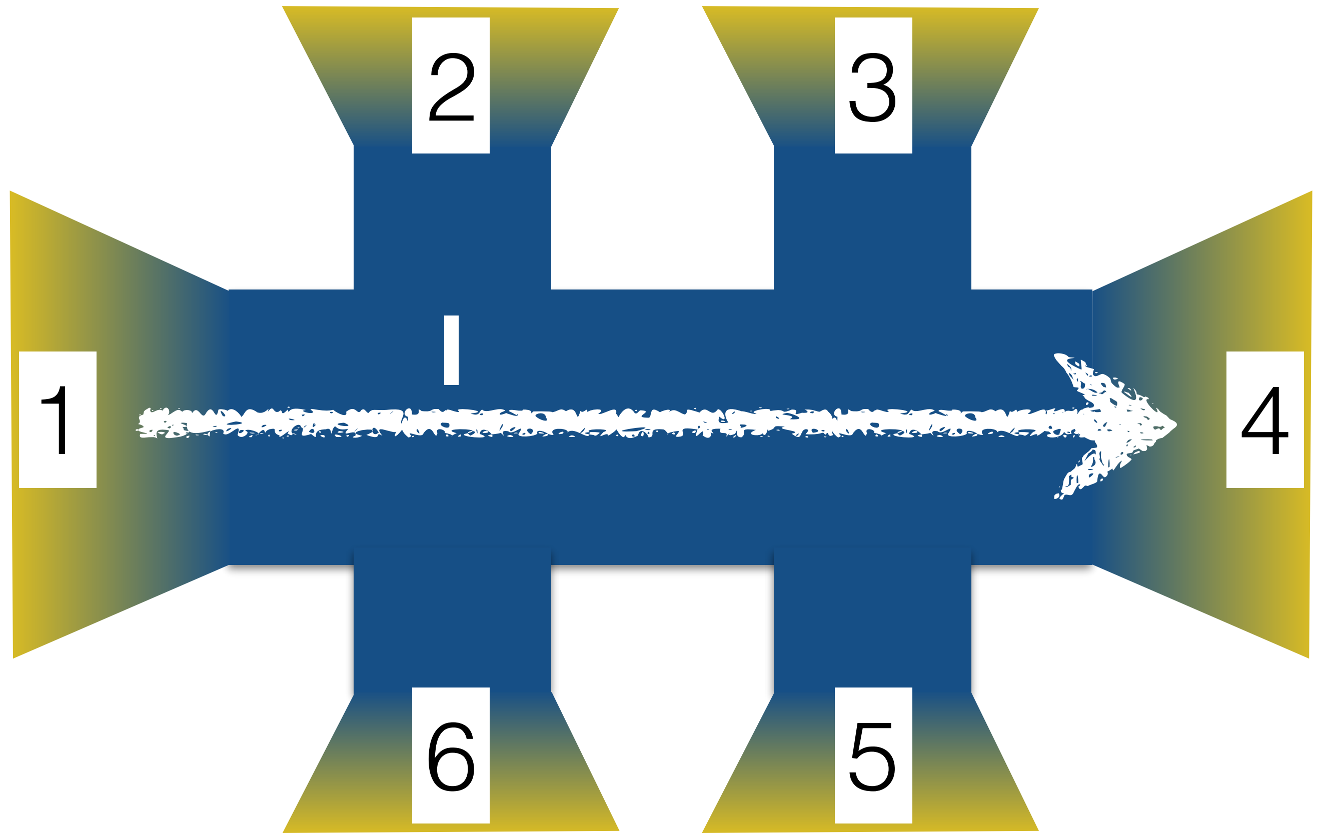
$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{pmatrix} = \frac{n(-e)}{h} \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mu_1 - \mu_6 \\ \mu_2 - \mu_6 \\ \mu_3 - \mu_6 \\ \mu_4 - \mu_6 \\ \mu_5 - \mu_6 \end{pmatrix} \Rightarrow \begin{pmatrix} \mu_1 - \mu_6 \\ \mu_2 - \mu_6 \\ \mu_3 - \mu_6 \\ \mu_4 - \mu_6 \\ \mu_5 - \mu_6 \end{pmatrix} = \frac{h}{n(-e)} \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{pmatrix}$$

# Transport and edge states

## Quantum Hall effect

- $n$ chiral edges modes:

$$\begin{pmatrix} \mu_1-\mu_6 \\ \mu_2-\mu_6 \\ \mu_3-\mu_6 \\ \mu_4-\mu_6 \\ \mu_5-\mu_6 \end{pmatrix} = \frac{h}{n(-e)} \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{pmatrix}$$




(A)

- Currents: $I_2 = I_3 = I_5 = I_6 = 0$ and $I_4 = -I_1 = I$

$$\begin{pmatrix} \mu_1-\mu_6 \\ \mu_2-\mu_6 \\ \mu_3-\mu_6 \\ \mu_4-\mu_6 \\ \mu_5-\mu_6 \end{pmatrix} = \frac{h}{n(-e)} \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -I \\ 0 \\ 0 \\ +I \\ 0 \end{pmatrix} = -\frac{hI}{ne} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow$ Only 2 possible conductances, $G_{\text{Hall}} = \dfrac{I}{V_6 - V_2} = G_{14,26} = G_{14,25} = n\dfrac{e^2}{h}$ and $G_{14,23} = G_{14,56} = 0$

# Transport and edge states

## Quantum Hall effect

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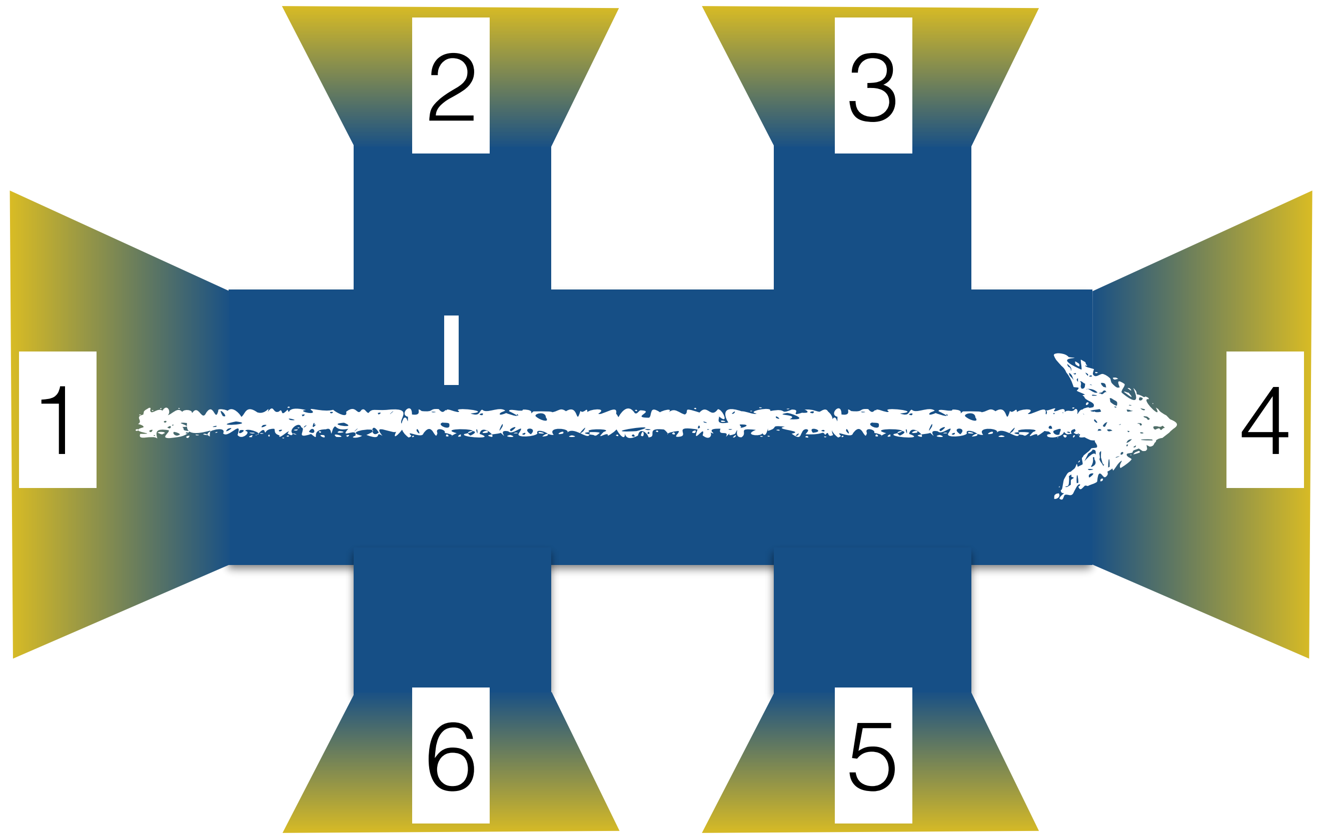

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- Currents: $I_2 = I_4 = I_5 = I_6 = 0$ and $I_3 = -I_1 = I$

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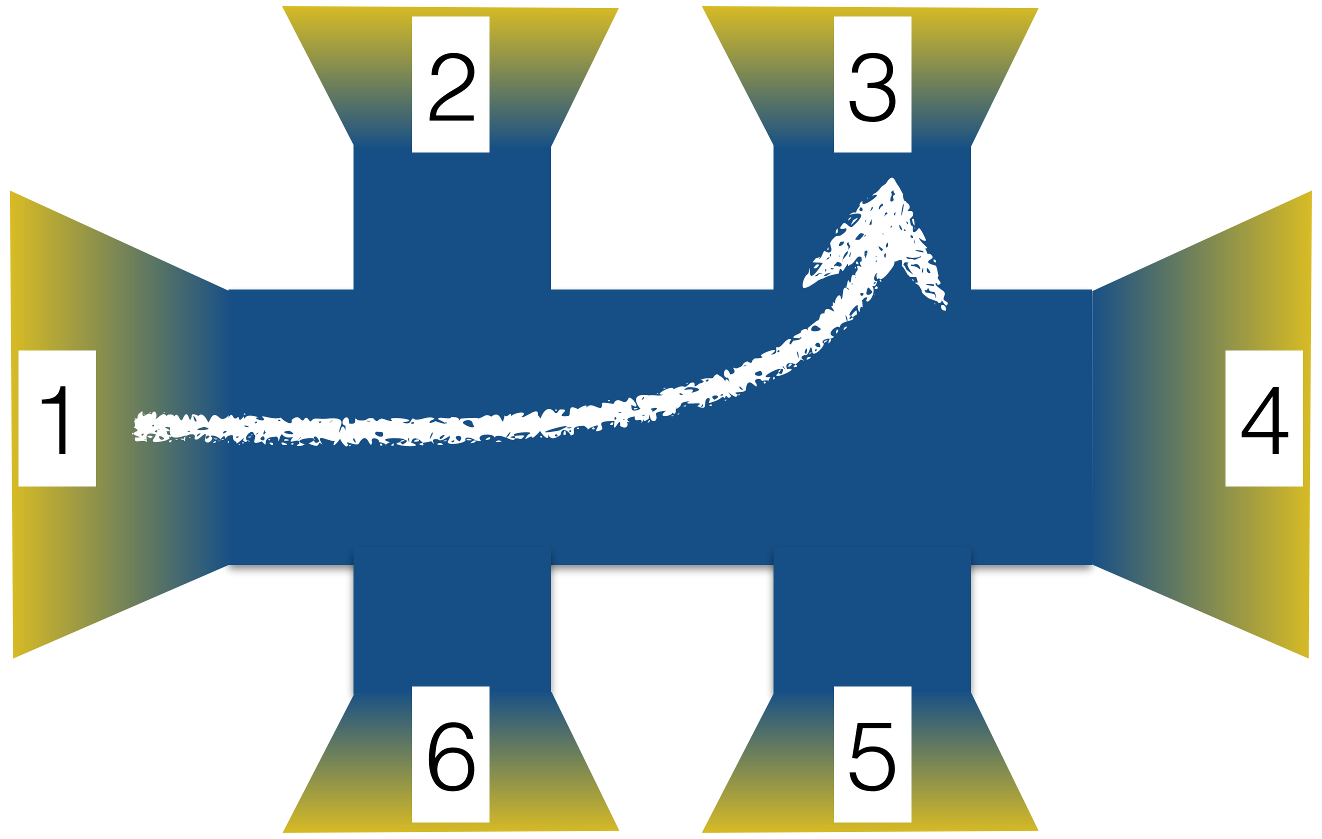
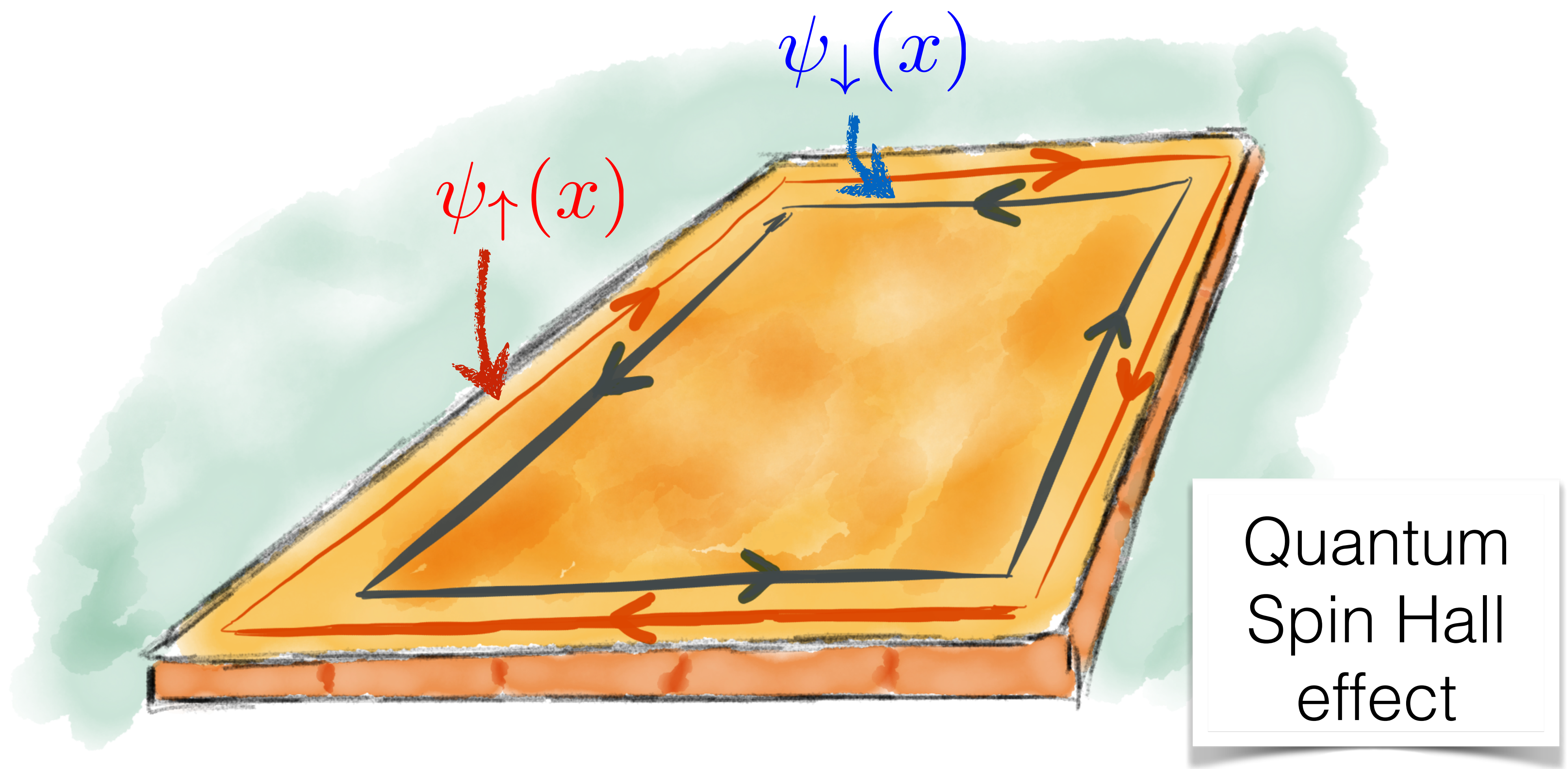



$\Rightarrow$ Again 2 equipotentials and 2 conductances, $G_{\text{Hall}} = G_{13,26} = n\frac{e^2}{h}$ and $G_{13,26} = 0$

(B)

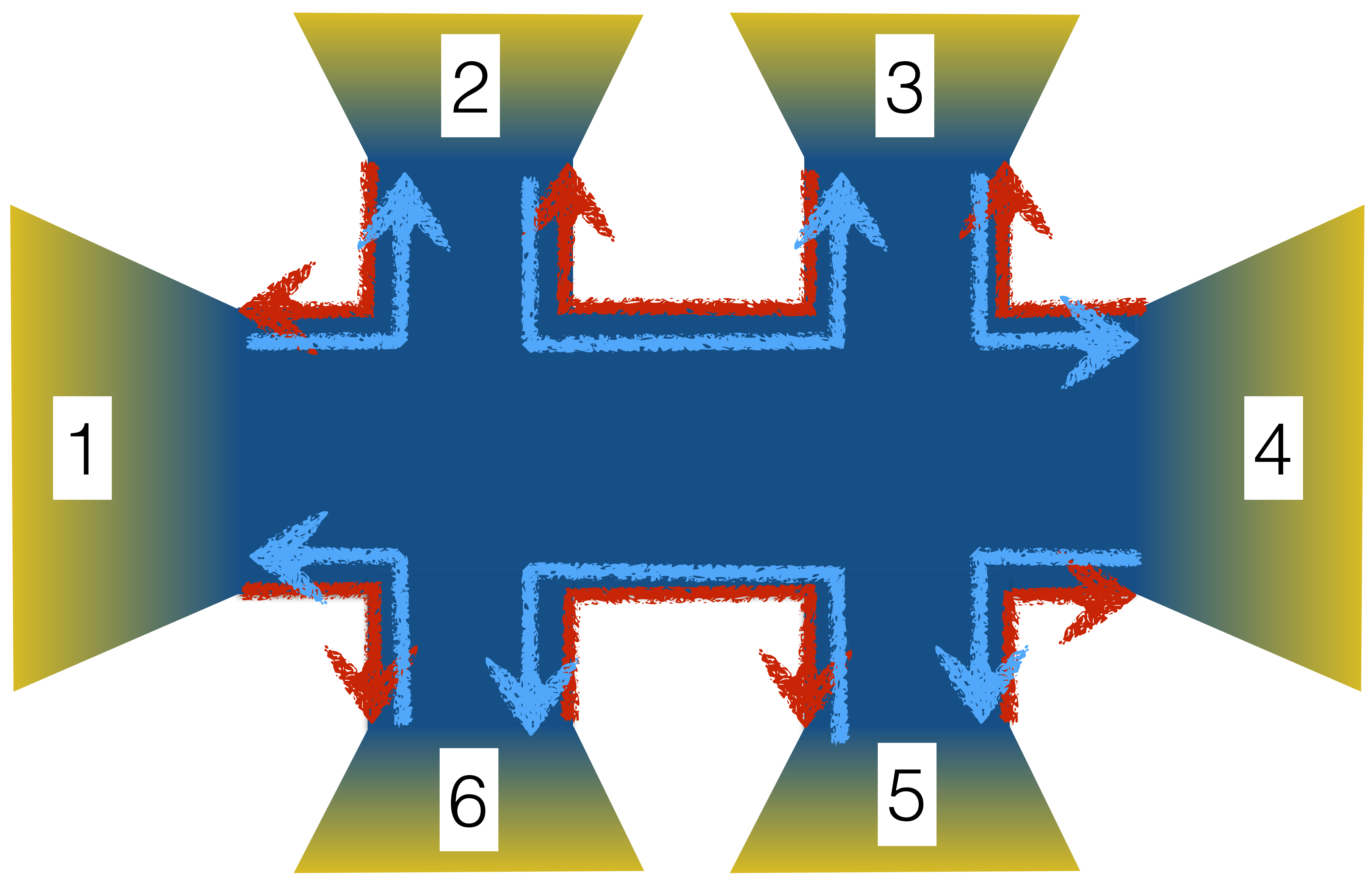
# Transport and edge states: Quantum Spin Hall effect

- **Landauer-Buttiker formalism**
  - Multi-contacts: $I_\alpha = \frac{-e}{h} \sum_{\beta \neq \alpha} T_{\alpha\beta} \left( \mu_\beta - \mu_\alpha \right)$




- **Quantum Spin Hall effect**
  - 2 counter propagating chiral edges modes:

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} = \frac{(-e)}{h} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix}$$



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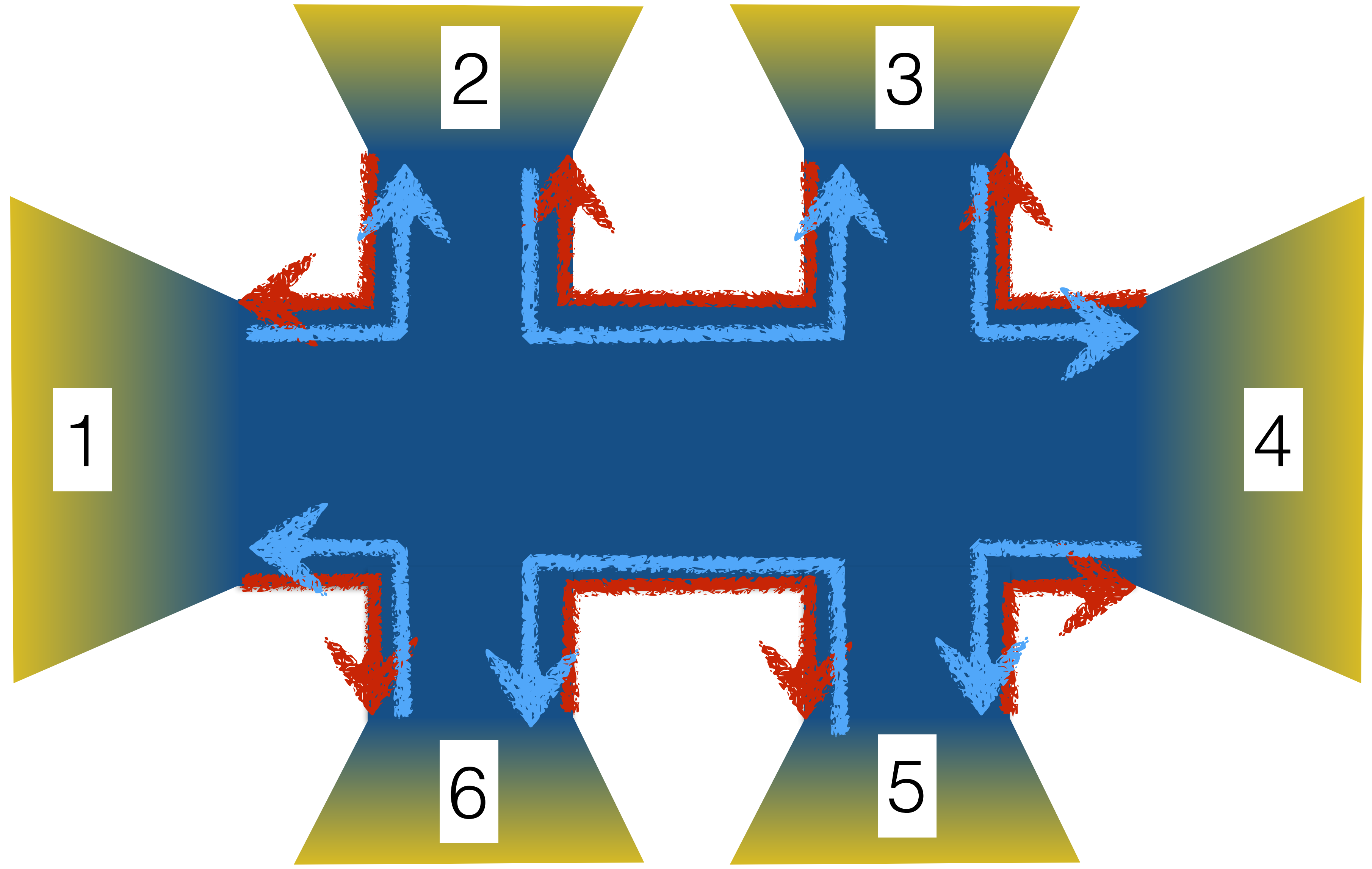



$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} = \frac{(-e)}{h} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix}$$

  - Current conservation: $\sum_\alpha I_\alpha = 0$ and definition of $\mu = 0$:

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{pmatrix} = \frac{(-e)}{h} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} \mu_1 - \mu_6 \\ \mu_2 - \mu_6 \\ \mu_3 - \mu_6 \\ \mu_4 - \mu_6 \\ \mu_5 - \mu_6 \end{pmatrix} \Rightarrow \begin{pmatrix} \mu_1 - \mu_6 \\ \mu_2 - \mu_6 \\ \mu_3 - \mu_6 \\ \mu_4 - \mu_6 \\ \mu_5 - \mu_6 \end{pmatrix} = \frac{-h}{6e} \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{pmatrix}$$

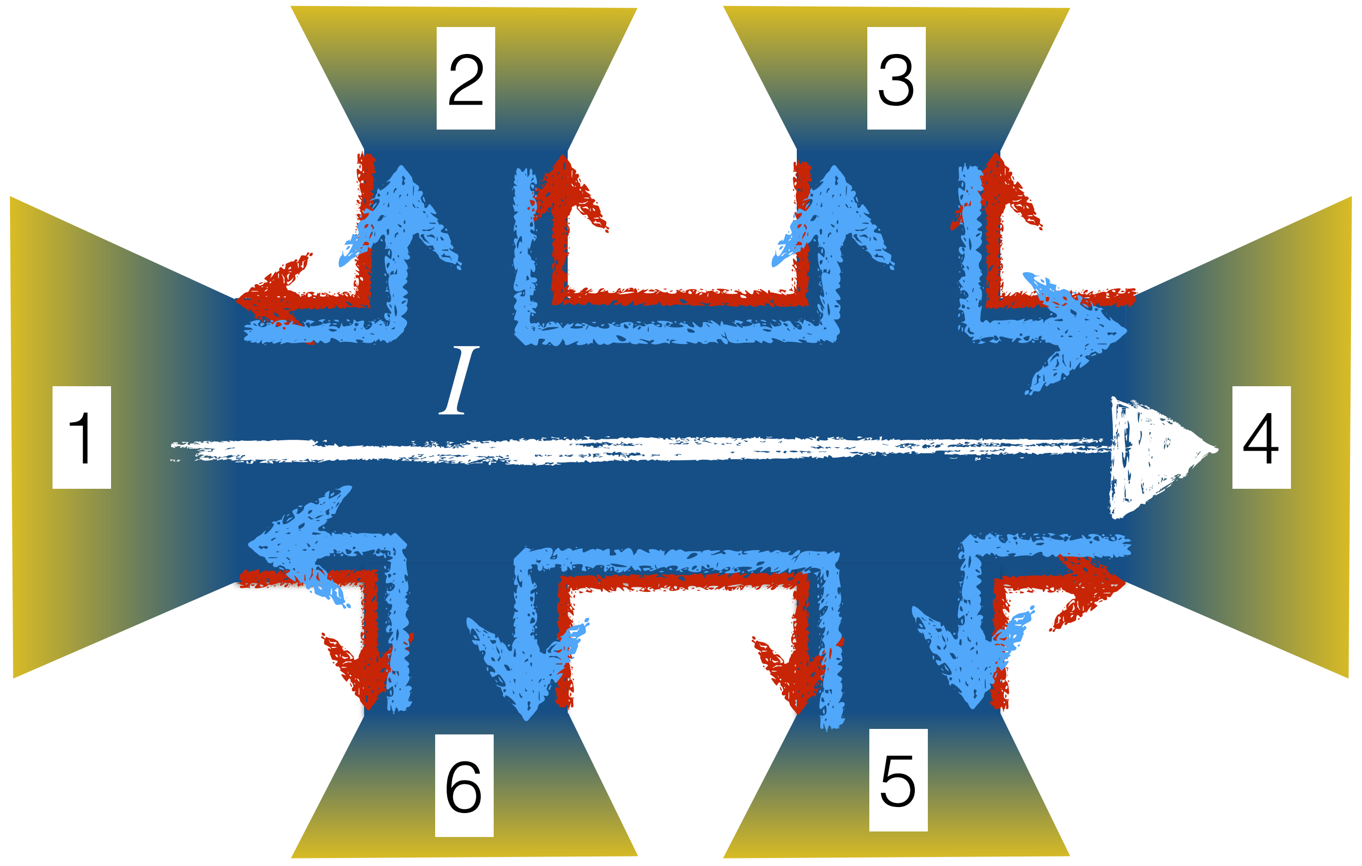
# Transport and edge states: Quantum Spin Hall effect

$$\begin{pmatrix}\mu_1-\mu_6\\ \mu_2-\mu_6\\ \mu_3-\mu_6\\ \mu_4-\mu_6\\ \mu_5-\mu_6\end{pmatrix} = \frac{-h}{6e}\begin{pmatrix}5&4&3&2&1\\4&8&6&4&2\\3&6&9&6&3\\2&4&6&8&4\\1&2&3&4&5\end{pmatrix}\begin{pmatrix}I_1\\I_2\\I_3\\I_4\\I_5\end{pmatrix}$$

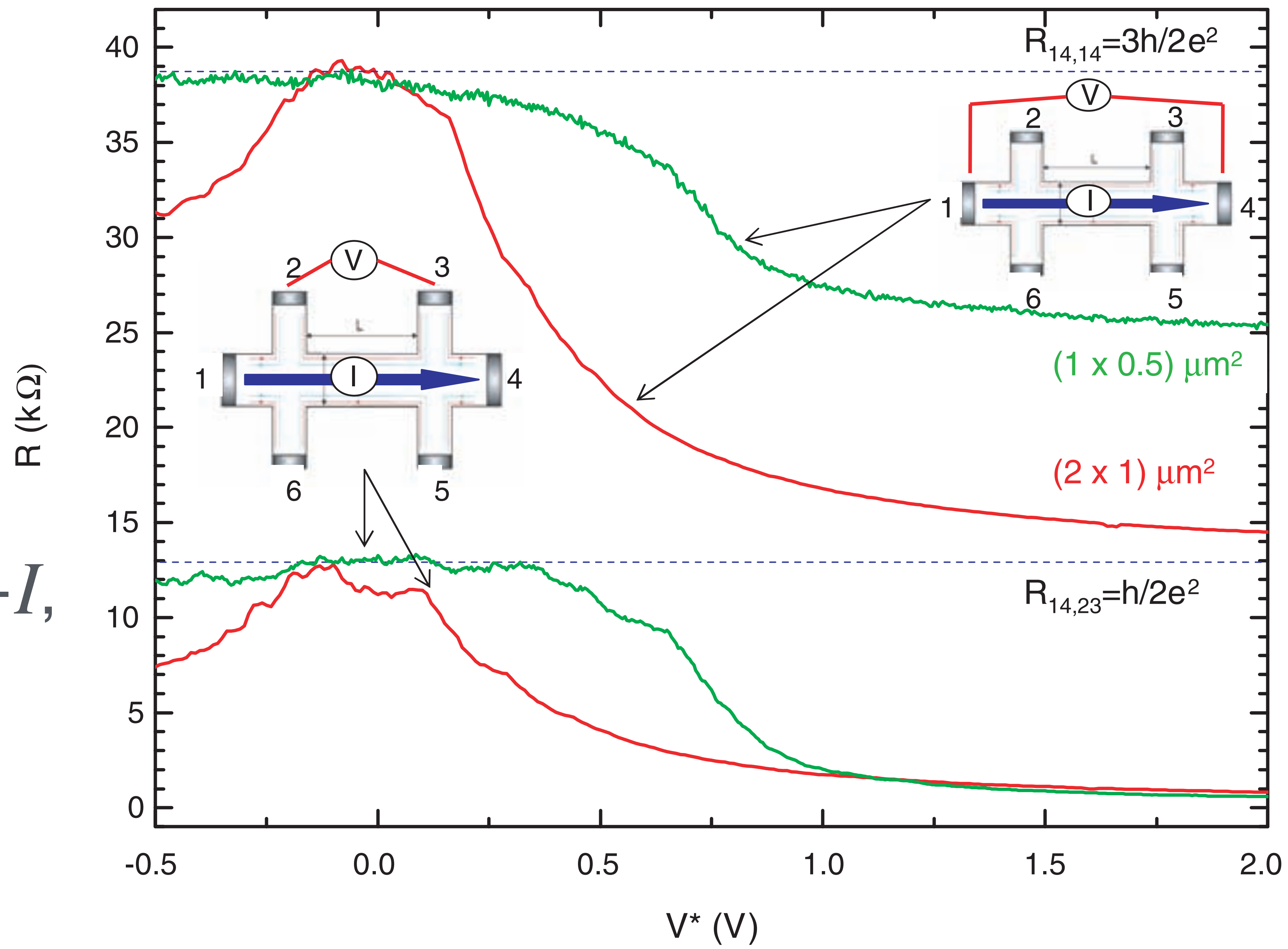
$$\begin{pmatrix}\mu_1-\mu_6\\ \mu_2-\mu_6\\ \mu_3-\mu_6\\ \mu_4-\mu_6\\ \mu_5-\mu_6\end{pmatrix} = \frac{-h}{6e}\begin{pmatrix}5&4&3&2&1\\4&8&6&4&2\\3&6&9&6&3\\2&4&6&8&4\\1&2&3&4&5\end{pmatrix}\begin{pmatrix}-I\\0\\0\\I\\0\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\\1\\\frac{1}{2}\end{pmatrix}\times\frac{-h}{e}I$$

$$\mu_4-\mu_1=\mu_4-\mu_6+\mu_6-\mu_1=\left(1-(-\frac{1}{2})\right)\frac{-h}{e}I=\frac{3}{2}\frac{-h}{e}I,$$

$$\Rightarrow R_{14,14}=\frac{V_4-V_1}{I}=\frac{3}{2}\frac{h}{e^2}$$




Roth et al., Science **325**, 294 (2009)



# Transport and edge states: Quantum Spin Hall effect

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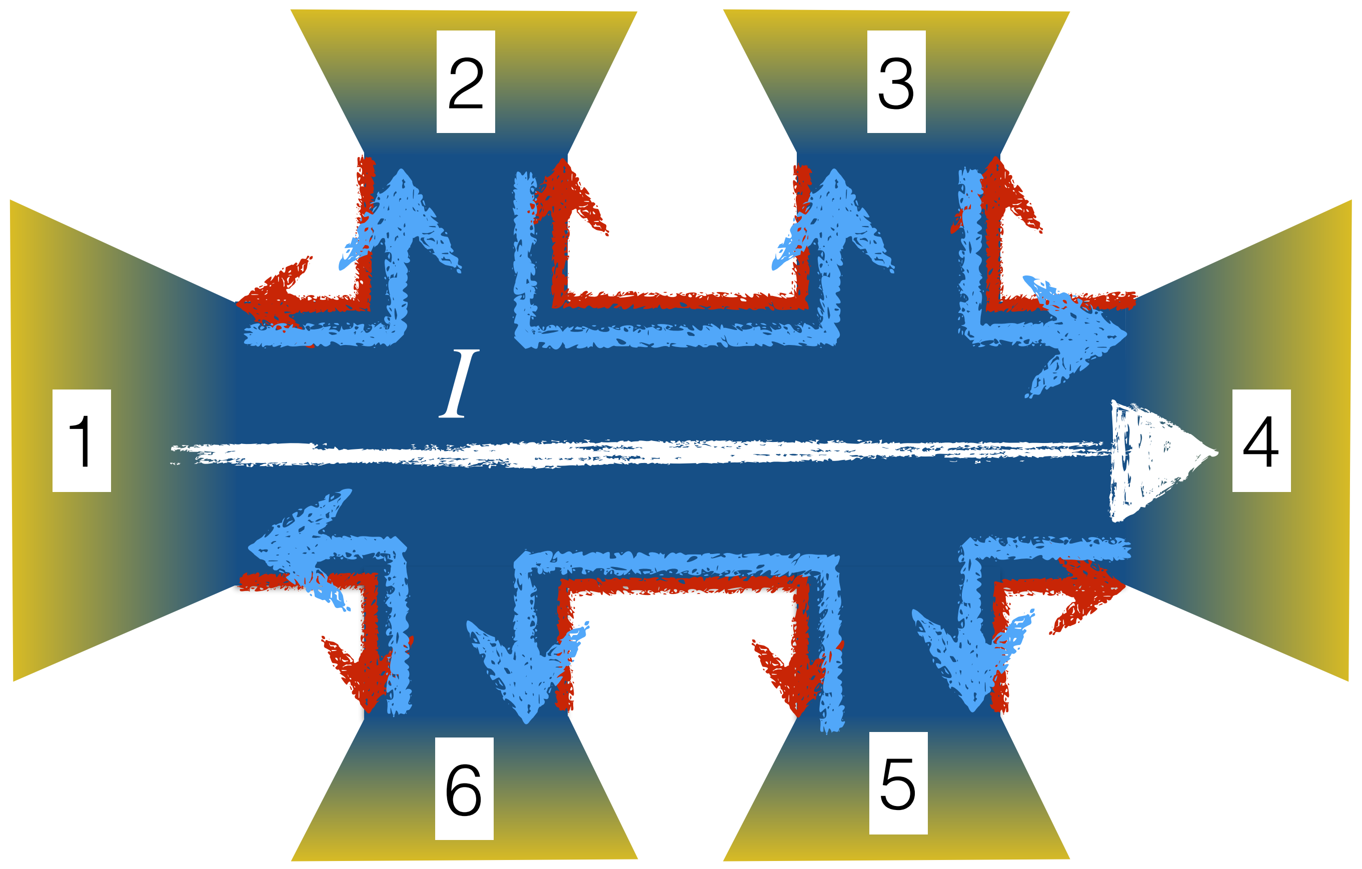
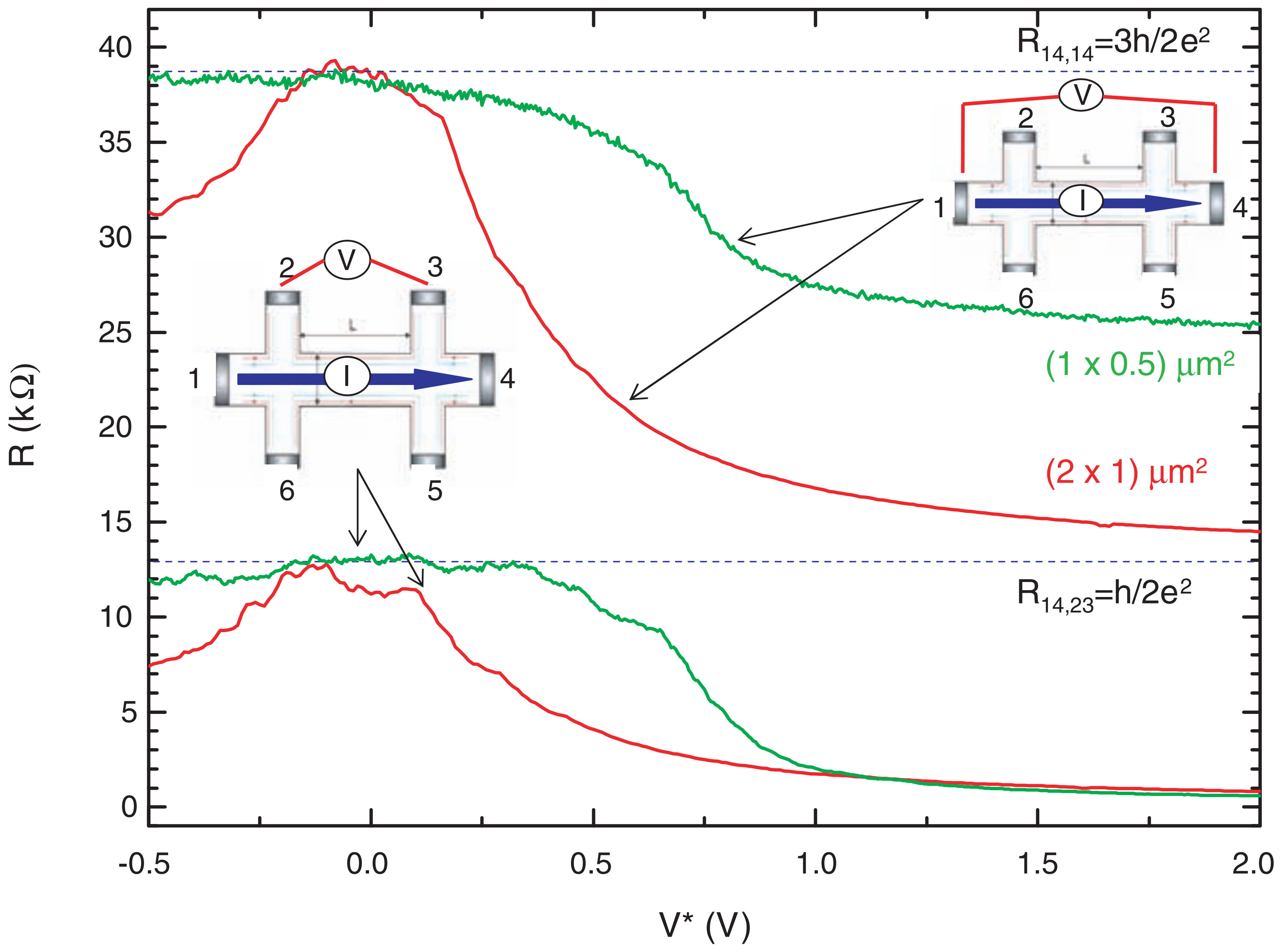
$$\begin{pmatrix}\mu_1-\mu_6\\ \mu_2-\mu_6\\ \mu_3-\mu_6\\ \mu_4-\mu_6\\ \mu_5-\mu_6\end{pmatrix} = \frac{-h}{6e}\begin{pmatrix}5&4&3&2&1\\4&8&6&4&2\\3&6&9&6&3\\2&4&6&8&4\\1&2&3&4&5\end{pmatrix}\begin{pmatrix}-I\\0\\0\\I\\0\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\\1\\\frac{1}{2}\end{pmatrix}\times\frac{-h}{e}I$$

$$\mu_3-\mu_2 = \mu_3-\mu_6+\mu_6-\mu_2 = \left(\frac{1}{2}-0\right)\frac{-h}{e}I = \frac{1}{2}\frac{-h}{e}I,$$

$$\Rightarrow R_{14,23} = \frac{I}{V_4-V_1} = \frac{1}{2}\frac{h}{e^2}$$

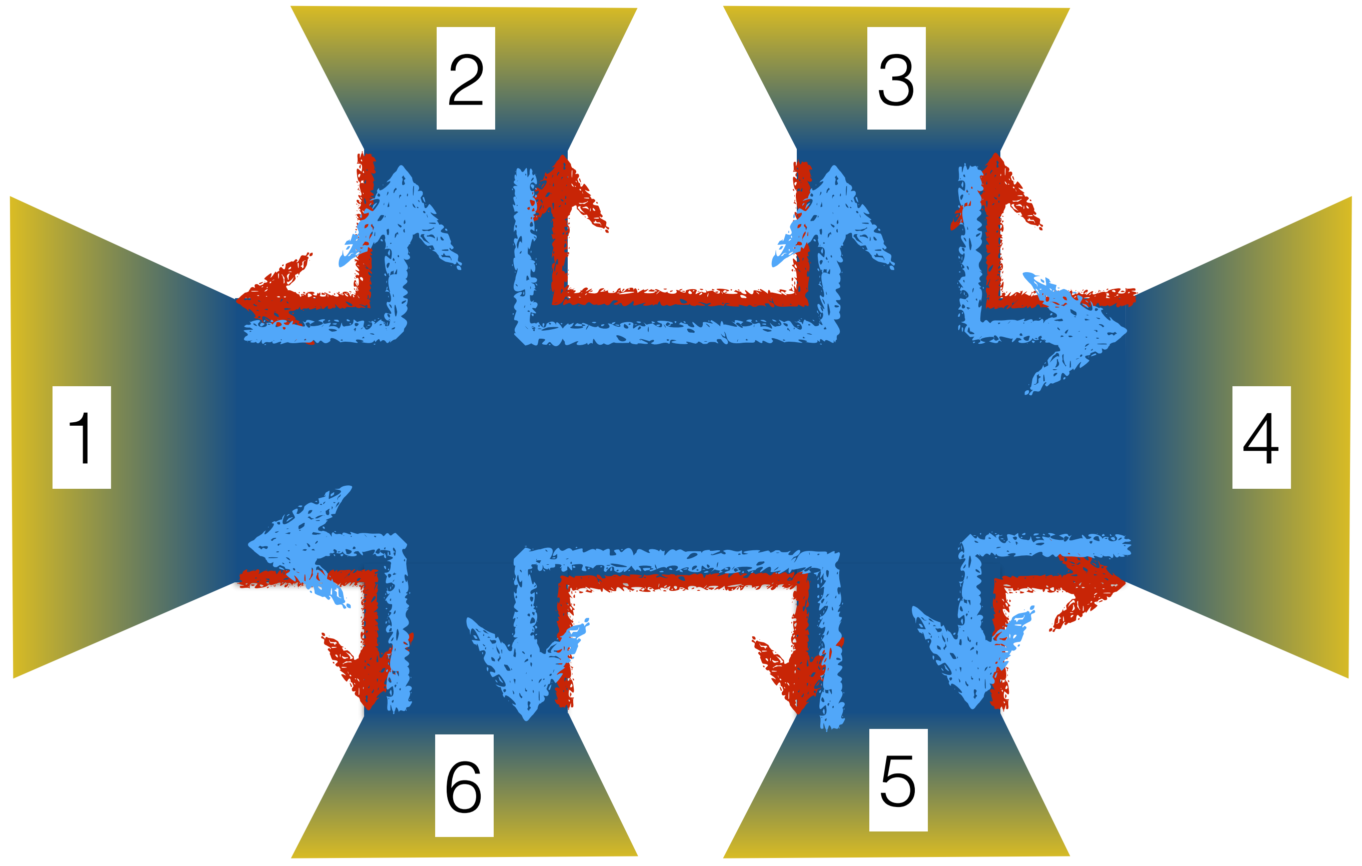


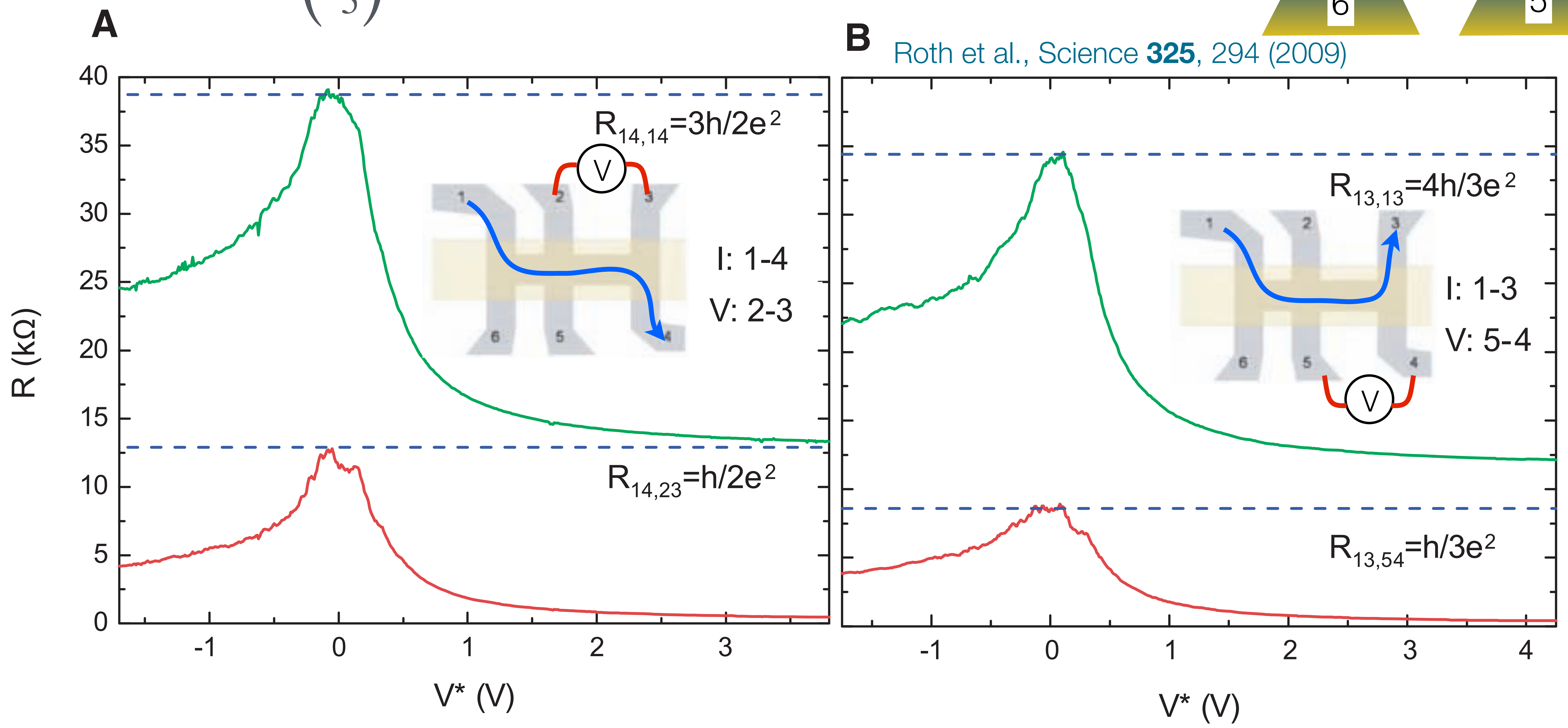

Roth et al., Science **325**, 294 (2009)



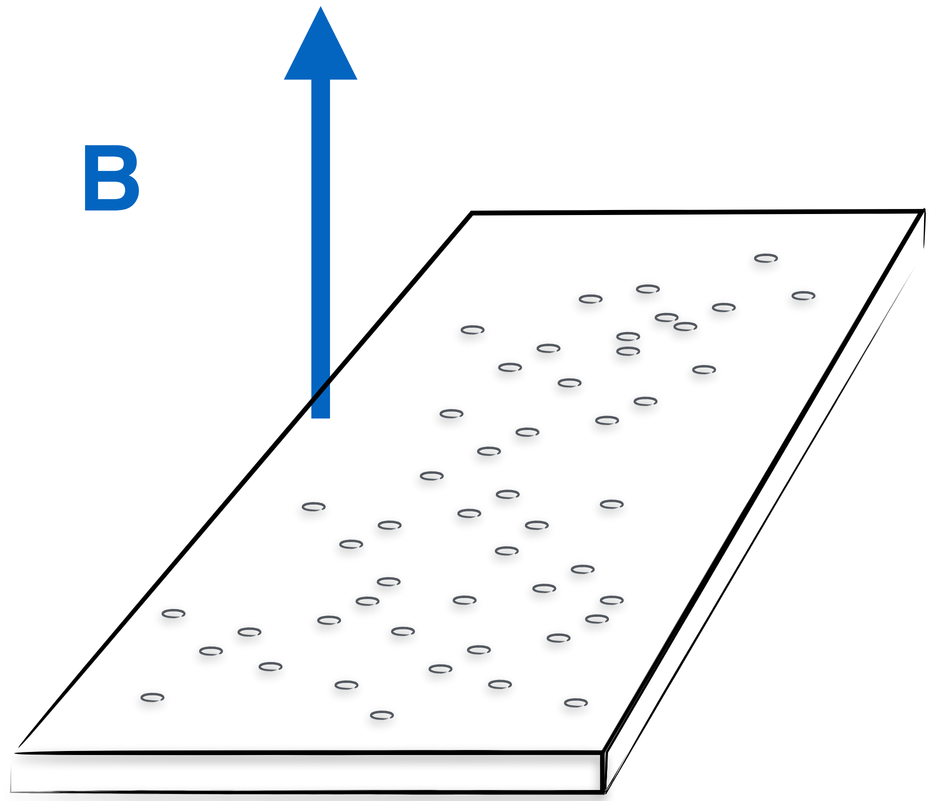
# Transport and edge states: Quantum Spin Hall effect

$$\begin{pmatrix} \mu_1 - \mu_6 \\ \mu_2 - \mu_6 \\ \mu_3 - \mu_6 \\ \mu_4 - \mu_6 \\ \mu_5 - \mu_6 \end{pmatrix} = \frac{-h}{6e} \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{pmatrix}$$



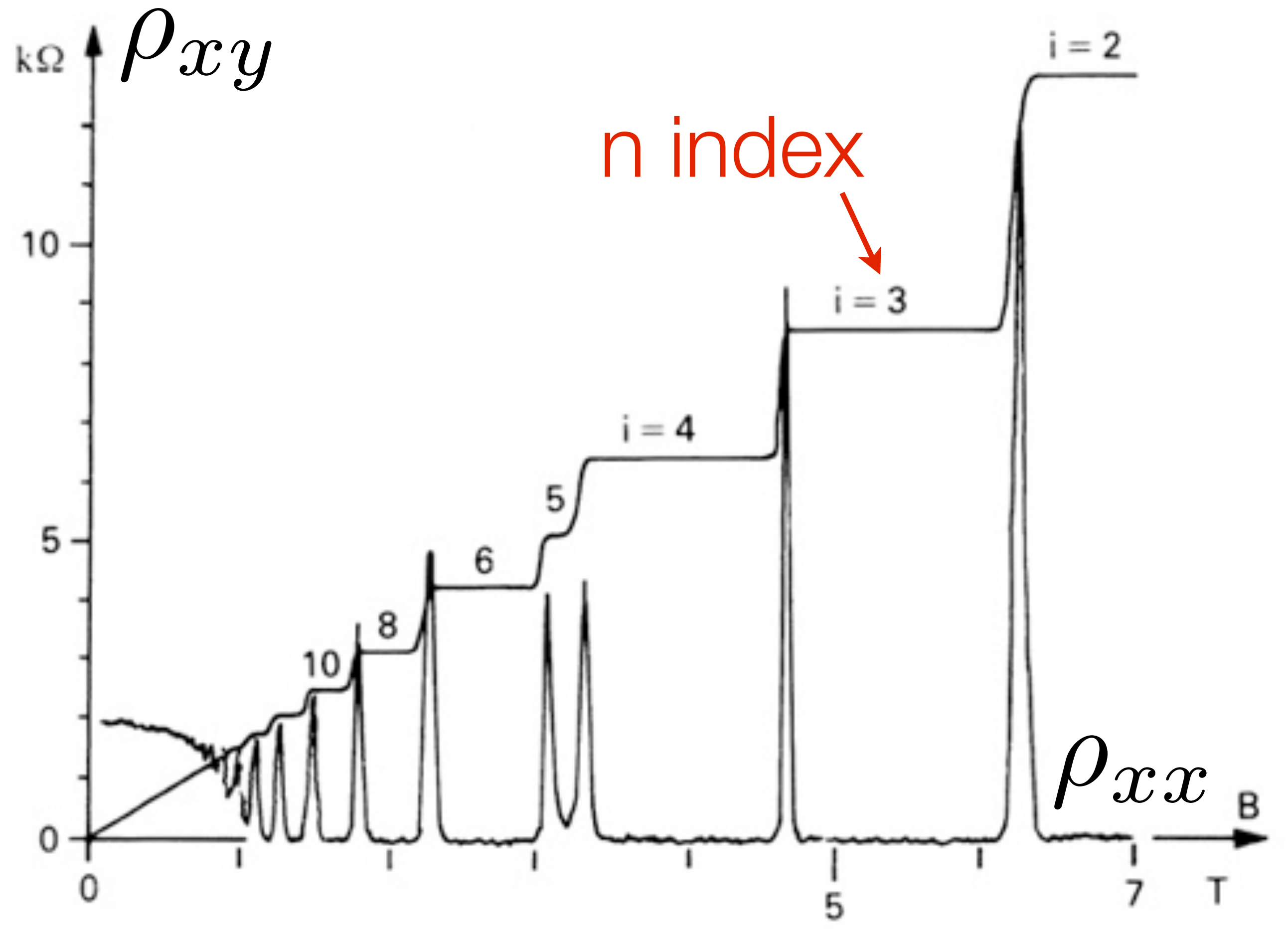
# Quantum Hall Effect and Chern Topological Insulator




2DEG (Heterojunction GaAs/AlGaAs)

$$\sigma_{xy} = n\frac{e^2}{h}$$

with high precision $(10^{-9})$




n is a topological invariant ↔ number of (robust) edge modes

Thouless *et al.*. PRL **49** (1982)

M. Büttiker, PRB **38** (1988)

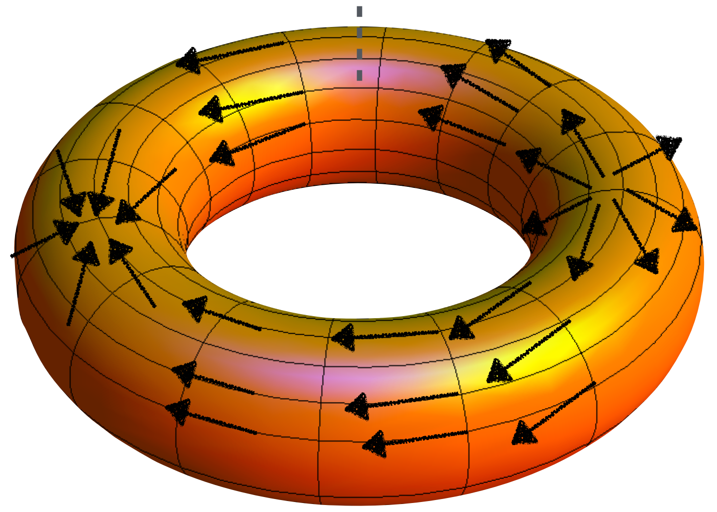
## Topology of Vector Bundle over the Brillouin/Boundary conditions Torus

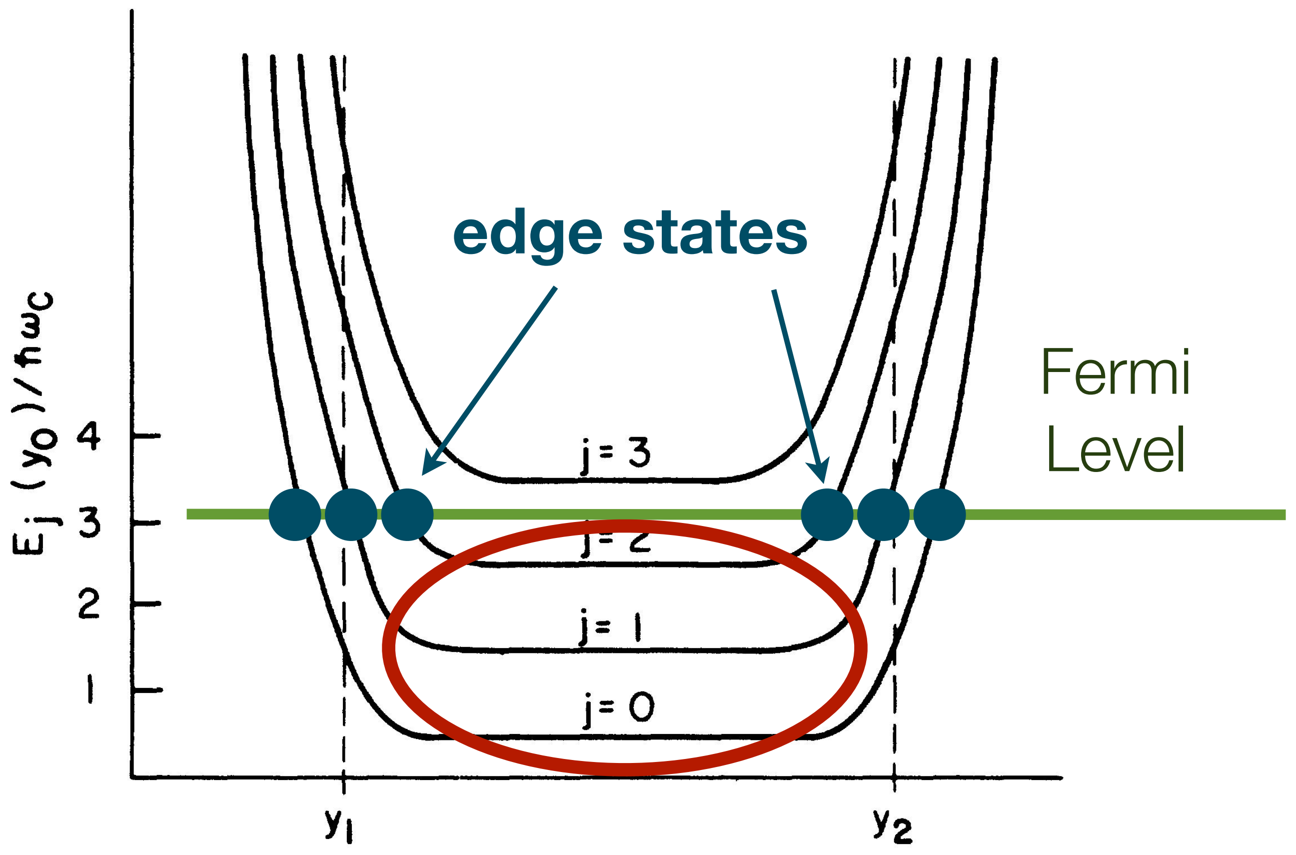
- Chern number $C_1 = \frac{1}{2\pi}\int dS\ F$

  with $F$ the Berry curvature

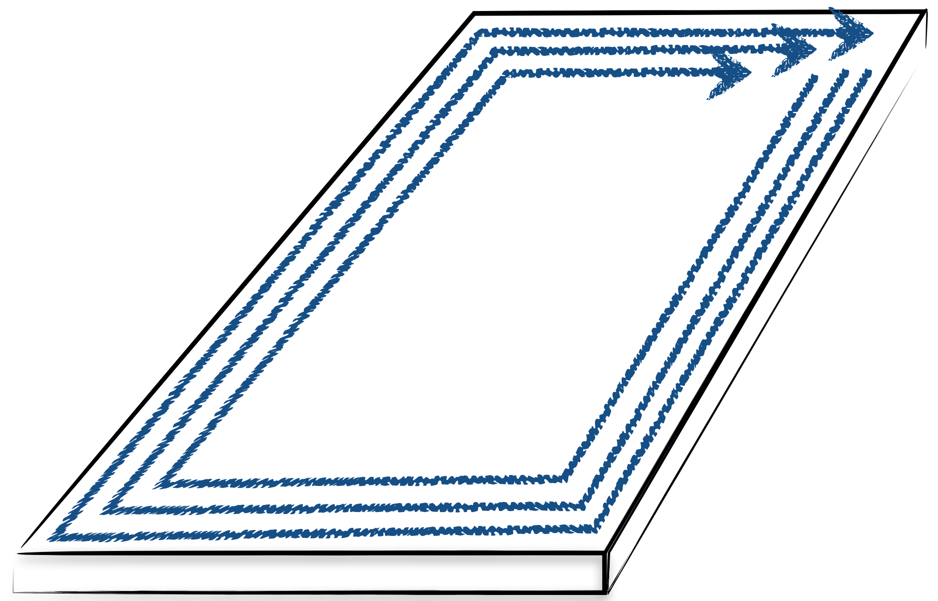
Filled Landau level

$E^1_k; |\psi^1_k\rangle$






Filled Landau Levels



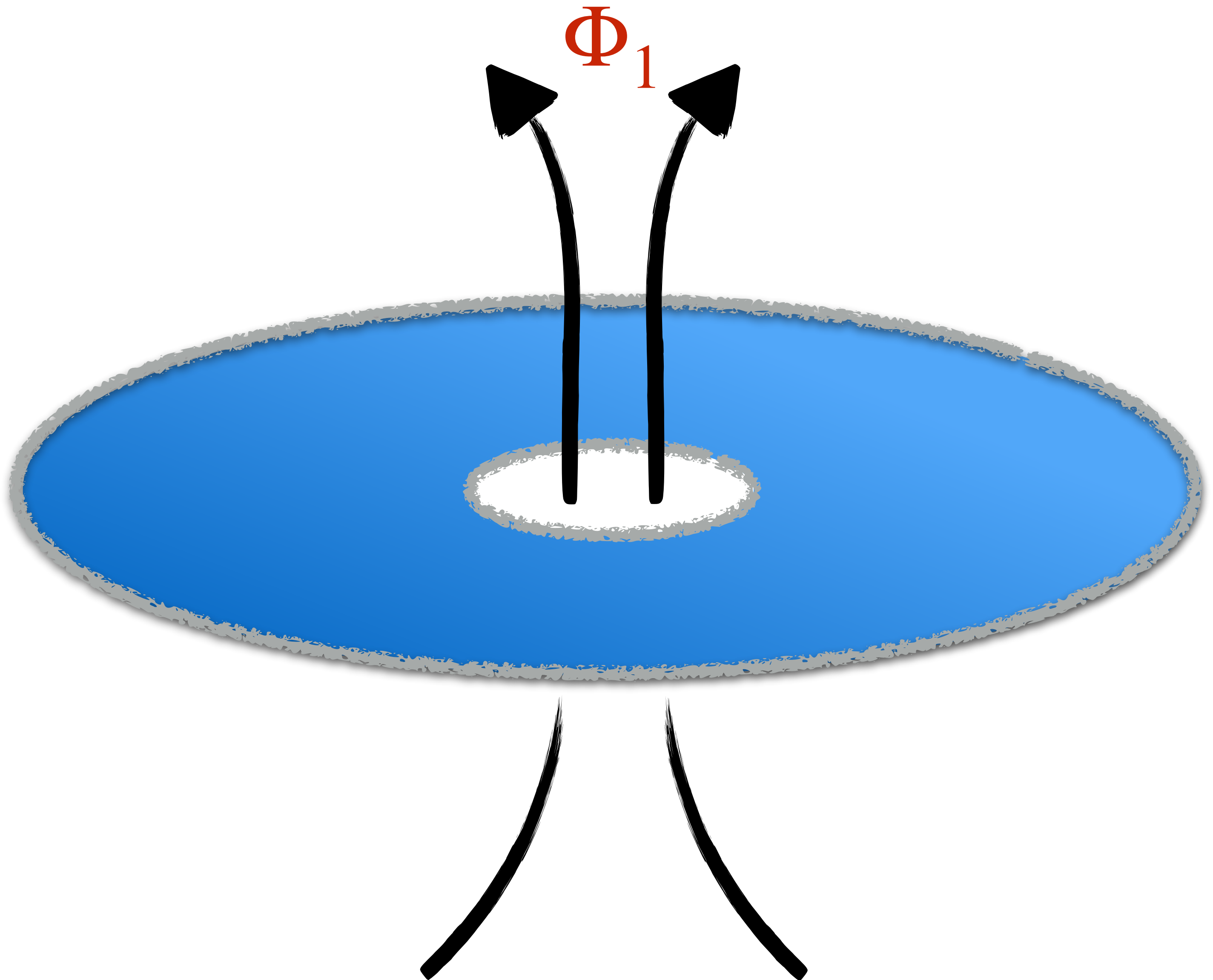
# Quantum Hall Effect and Chern Topological Insulator

- **Kubo formula (T=0)**

$$\sigma_{xy} = \frac{i\hbar}{L_x L_y} \sum_{m \neq 0} \frac{1}{(E_m - E_0)^2} \left[ \langle \Psi_0 | \hat{J}_x | \Psi_m \rangle \langle \Psi_m | \hat{J}_y | \Psi_0 \rangle - \langle \Psi_0 | \hat{J}_y | \Psi_m \rangle \langle \Psi_m | \hat{J}_x | \Psi_0 \rangle \right]$$

Many body state

Current $\hat{J}_\alpha(\mathbf{r}) = \dfrac{\delta H}{\delta A_\alpha(\mathbf{r})}$




Aharonov-Bohm flux

➡ Dephasing by $2\pi \dfrac{\Phi_1}{\phi_0}$ around flux

➡ Vector potential $A_r = \Phi_1 / L$

# Quantum Hall Effect and Chern Topological Insulator

Q. Niu, D. J. Thouless, Y.-S. Wu (1985)
J. E. Avron, R. Seiler, and L. G. Yaffe (1987)

J. Luneau, C. Dutreix, Q. Ficheux, P. Delplace, B. Douçot, B. Huard, D. Carpentier, Phys. Rev. Research 4, 013169 (2022)

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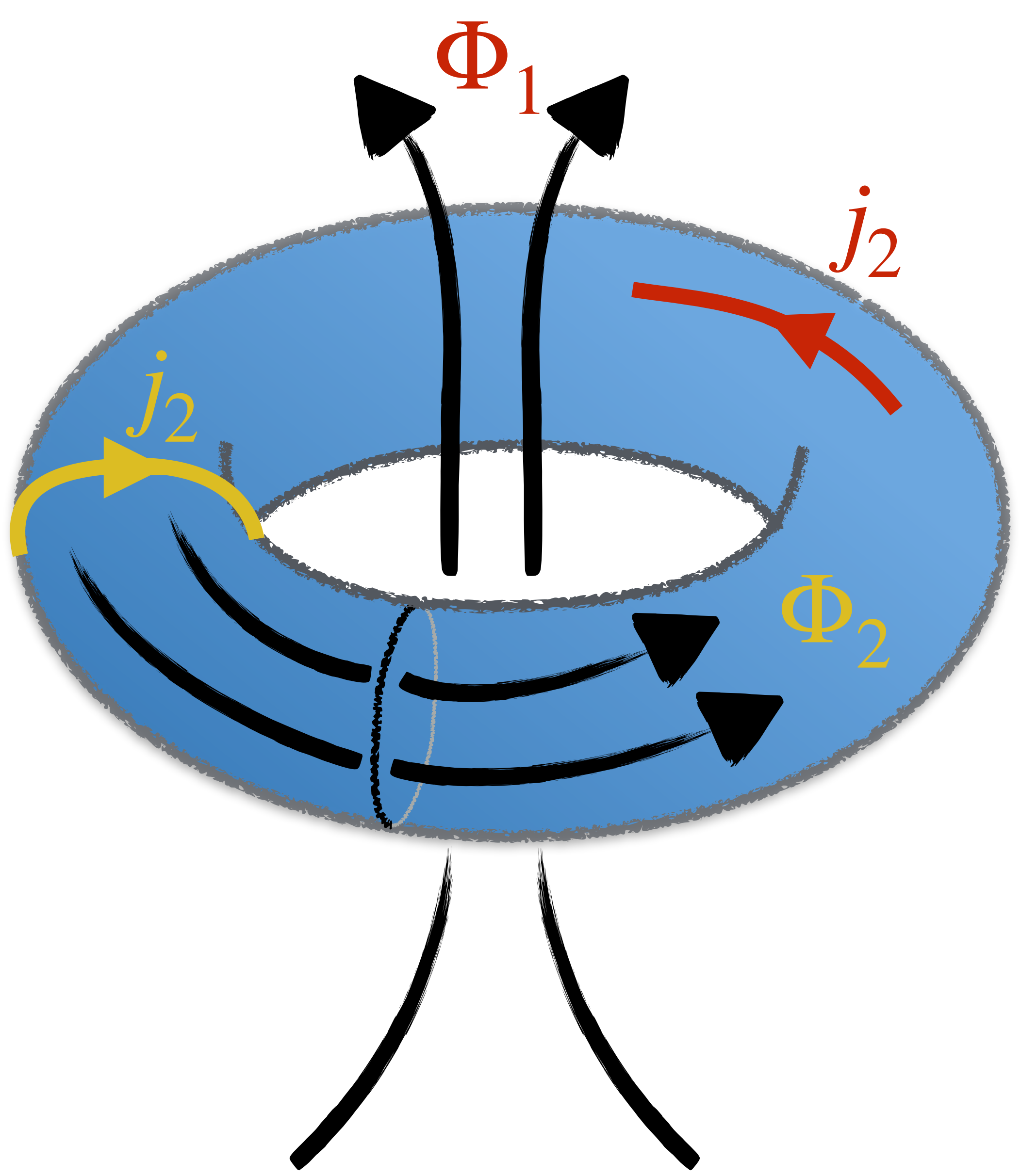
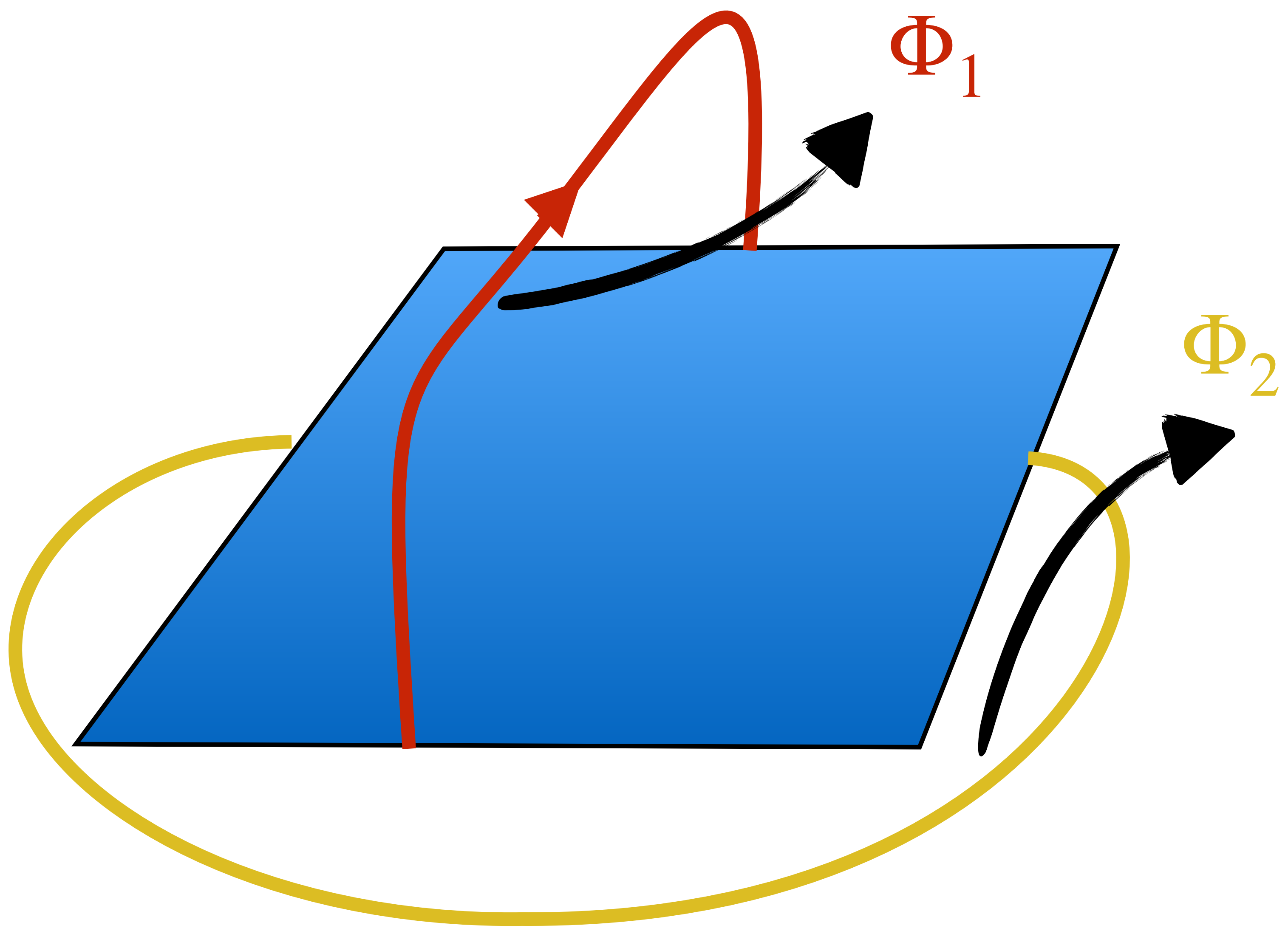
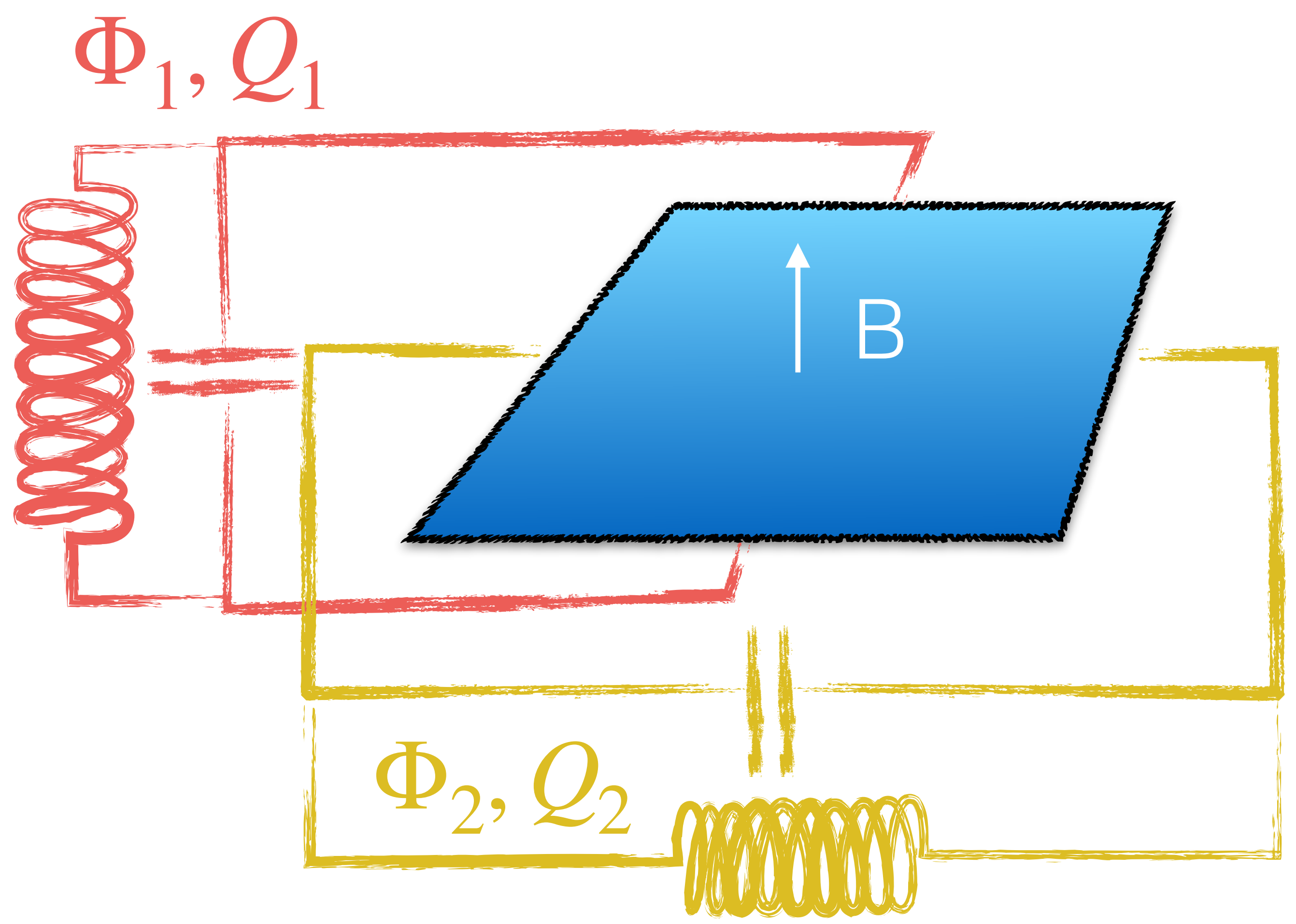
Current $\hat{J}_\alpha(\mathbf{r}) = \dfrac{\delta H}{\delta A_\alpha(\mathbf{r})} = L_\alpha \dfrac{\partial H}{\partial \Phi_\alpha}$

Aharonov-Bohm flux: $A_\alpha = \Phi_\alpha / L_\alpha$ (homogeneous)



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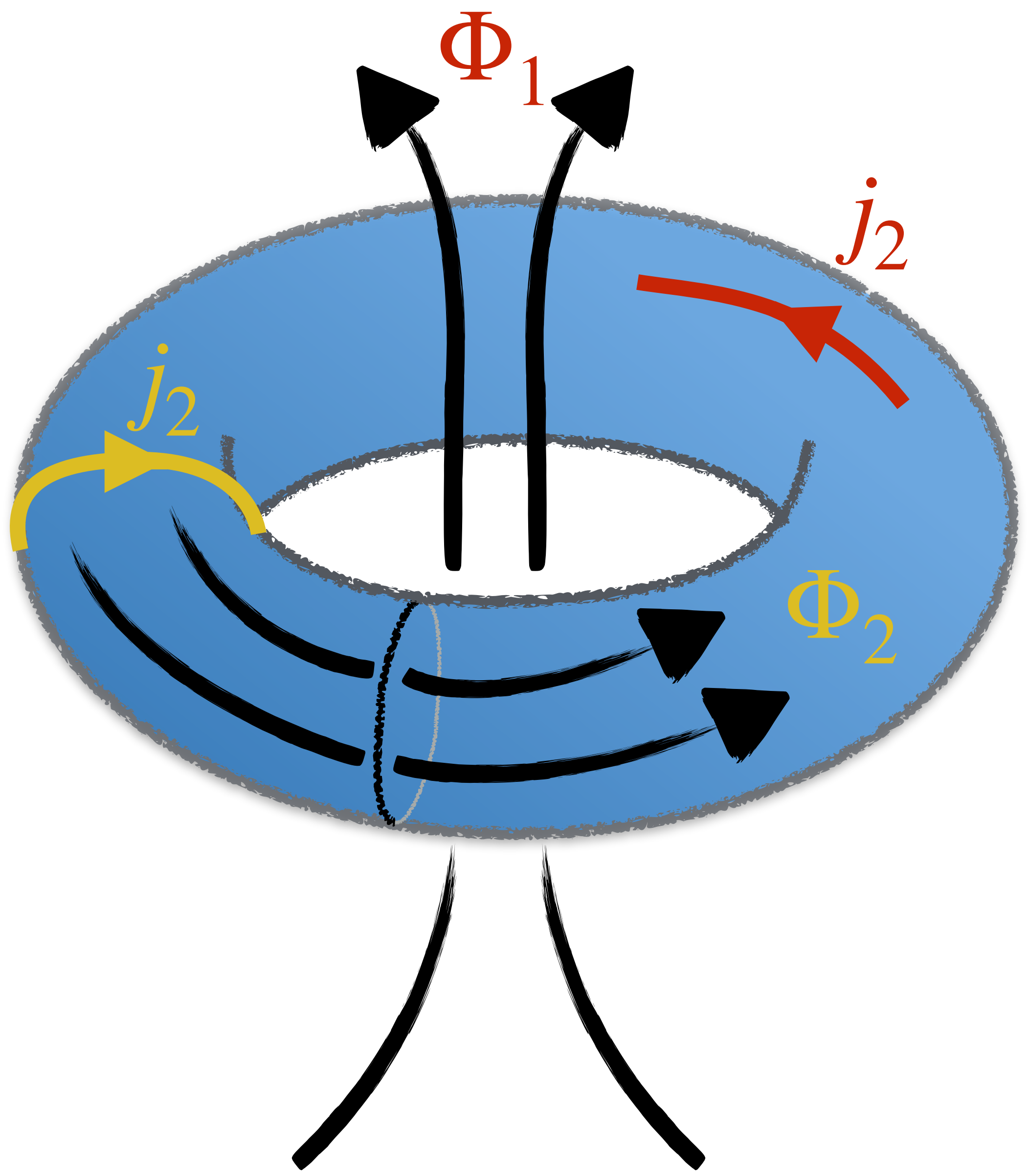


Many-body state $\Psi$ (generalized boundary condition)

$$\Psi(x_i + L_x, y_i) = e^{i2\pi\frac{\Phi_1}{\phi_0}} \Psi(x_i, y_i) \; ; \; \Psi(x_i, y_i + L_y) = e^{i2\pi\frac{\Phi_2}{\phi_0}} \Psi(x_i, y_i)$$

$\Rightarrow 2\pi$ periodicity of $\Psi$

# Quantum Hall Effect and Chern Topological Insulator

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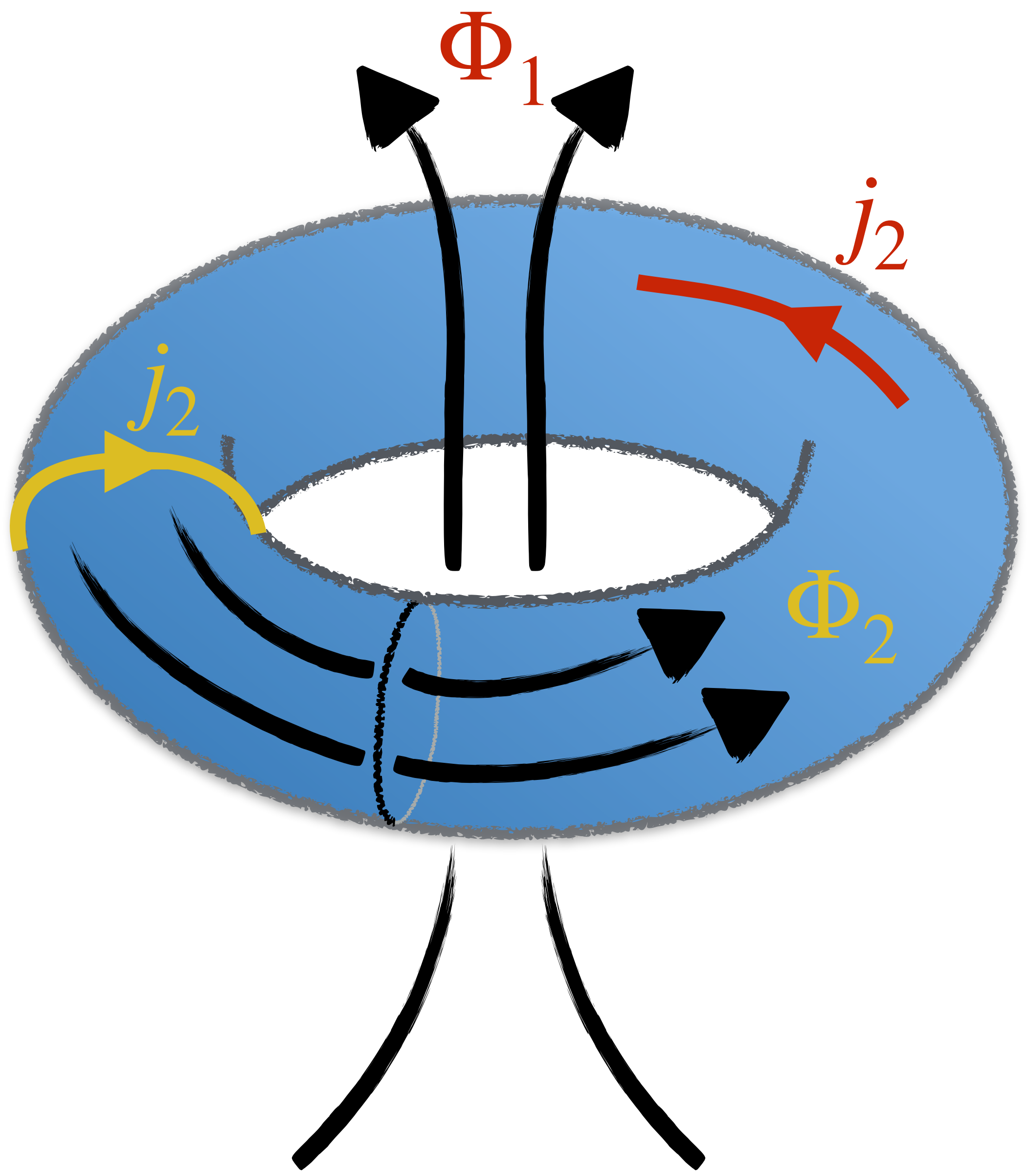


$$\sigma_{xy} = i\hbar \sum_{m\neq 0} \frac{1}{(E_m - E_0)^2} \left[ \langle \Psi_0 | \frac{\partial H}{\partial \Phi_x} | \Psi_m \rangle \langle \Psi_m | \frac{\partial H}{\partial \Phi_y} | \Psi_0 \rangle - \langle \Psi_0 | \frac{\partial H}{\partial \Phi_y} | \Psi_m \rangle \langle \Psi_m | \frac{\partial H}{\partial \Phi_x} | \Psi_0 \rangle \right]$$

$$= \hbar \mathscr{B}(\Phi_1, \Phi_2)$$ → Berry curvature of $\Psi(\Phi_1, \Phi_2)$

Should be independent on boundary condition:

$$\sigma_{xy} = \frac{1}{\phi_0^2} \int d^2\mathbf{\Phi}\ \sigma_{xy}(\mathbf{\Phi}) = \frac{e^2}{h^2} \frac{h}{2\pi} \int d^2\mathbf{\Phi}\ \mathscr{B}(\mathbf{\Phi}) = \frac{e^2}{h} \mathscr{C}$$

with $\mathscr{C}$: Chern number of $\Psi(\Phi_1, \Phi_2)$

# Quantum Hall Effect and Chern Topological Insulator

## Electrodynamics of an insulator

- Terms invariant under gauge transformations $A_\mu \to A_\mu + \partial_\mu \chi(\mathbf{r})$: $B_j \chi_{ij} B_j$ , $E_i p_{ij} E_j$
- Standard Maxwell Lagrangian (isotropic): $\mathscr{L}_0 = \frac{\epsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2 - \rho\phi + \mathbf{j}\,.\,\mathbf{A}$
- Action $\mathscr{S} = \int d^2\mathbf{r}dt\ \mathscr{L}$
- $\frac{\delta \mathscr{S}}{\delta \phi} = 0 = -\rho + \epsilon_0\ \nabla\,.\,\mathbf{E}$
- $\frac{\delta \mathscr{S}}{\delta \mathbf{A}} = 0 = \mathbf{j} + \epsilon_0\ \dot{\mathbf{E}} - \frac{1}{\mu_0}\nabla \times \mathbf{B}$

## Topological electrodynamics in d=2:

- In d=2, extra term (Chern-Simons) invariant under gauge transformations: $\frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$

  With $A_\mu = (A_0, -\mathbf{A})$
- Under a gauge transformation $(A_\mu \to A_\mu + \partial_\mu \chi(\mathbf{r}))$ :

  $\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \to \epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \epsilon^{\mu\nu\lambda}\partial_\mu\chi\partial_\nu A_\lambda$

  with a modified action

  $\delta\mathscr{S} = \frac{\kappa}{4\pi}\int d^2\mathbf{r}dt\ \epsilon^{\mu\nu\lambda}\partial_\mu\left(\chi\partial_\nu A_\lambda\right)$

  Boundary term !

# Quantum Hall Effect and Chern Topological Insulator

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  - In d=2, extra term (Chern-Simons) invariant under gauge transformations: $\frac{\kappa}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu \partial_\nu A_\lambda$

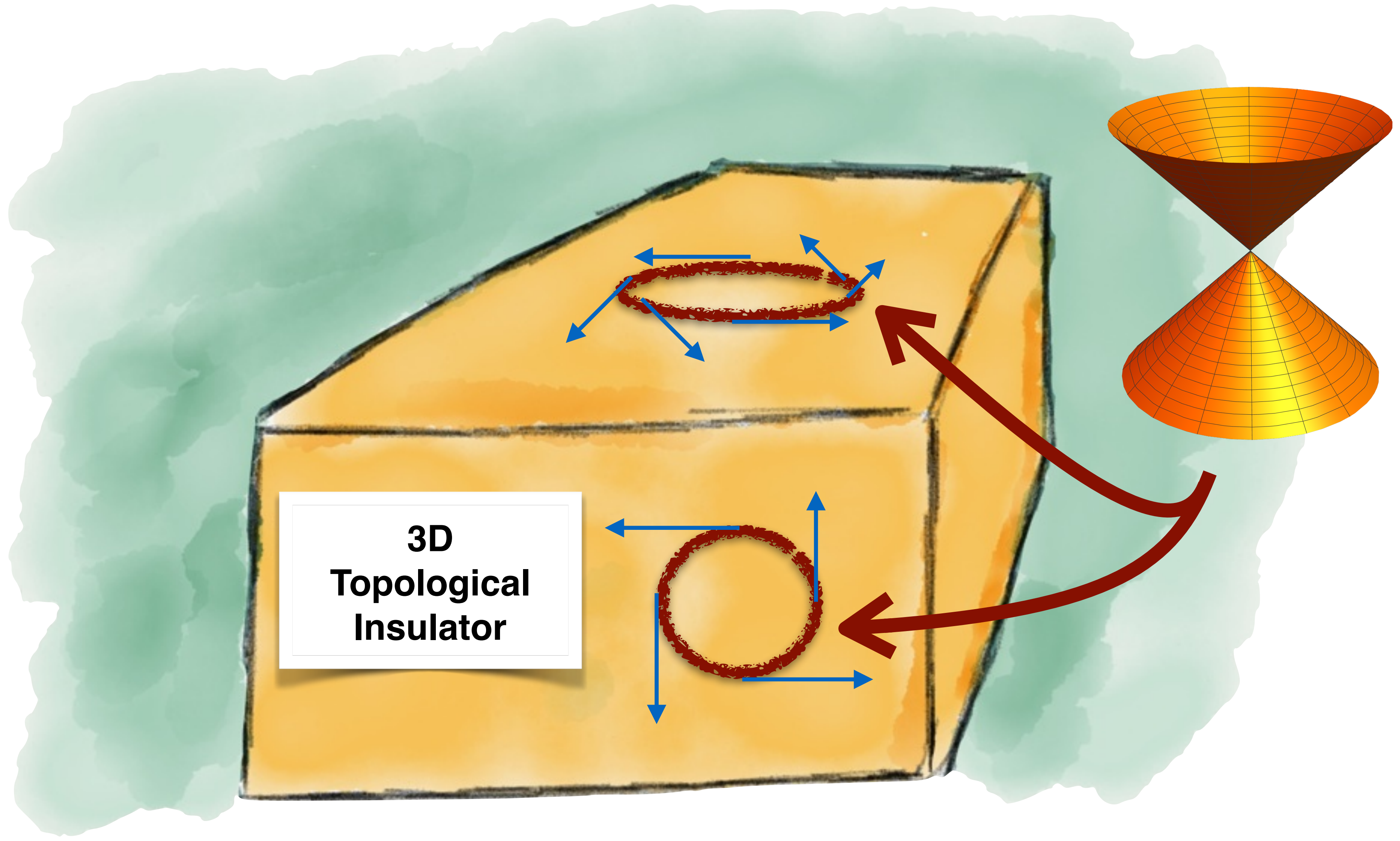
    With $A_\mu = (A_0, -\mathbf{A})$
  - Associated current:

    $$j_i = -\frac{\delta \mathscr{S}}{\delta A_i} = \frac{\kappa}{2\pi}\epsilon_{ij}\partial_0 A_j = \frac{\kappa}{2\pi}\epsilon_{ij}E_j$$
  - Quantized Hall conductivity

    $$\sigma_{xy} = \frac{\kappa}{2\pi}\frac{e^2}{\hbar} = \kappa\frac{e^2}{h}$$

# Outline

- **2D topological insulators**
  - Edge states versus bulk: the Quantum Hall Effect example
  - Modified electrodynamics and quantized Hall response in 2D
- **3D topological insulators**
  - Surface properties
  - Modified electrodynamics in 3D: magneto-electric effects
  - (A choice of some) consequences
- **3D topological semimetals**
  - Recap of topological properties: anomalous Hall effect
  - Modified electrodynamics
  - Non-reciprocity of electrodynamics: consequences on optical and thermal properties

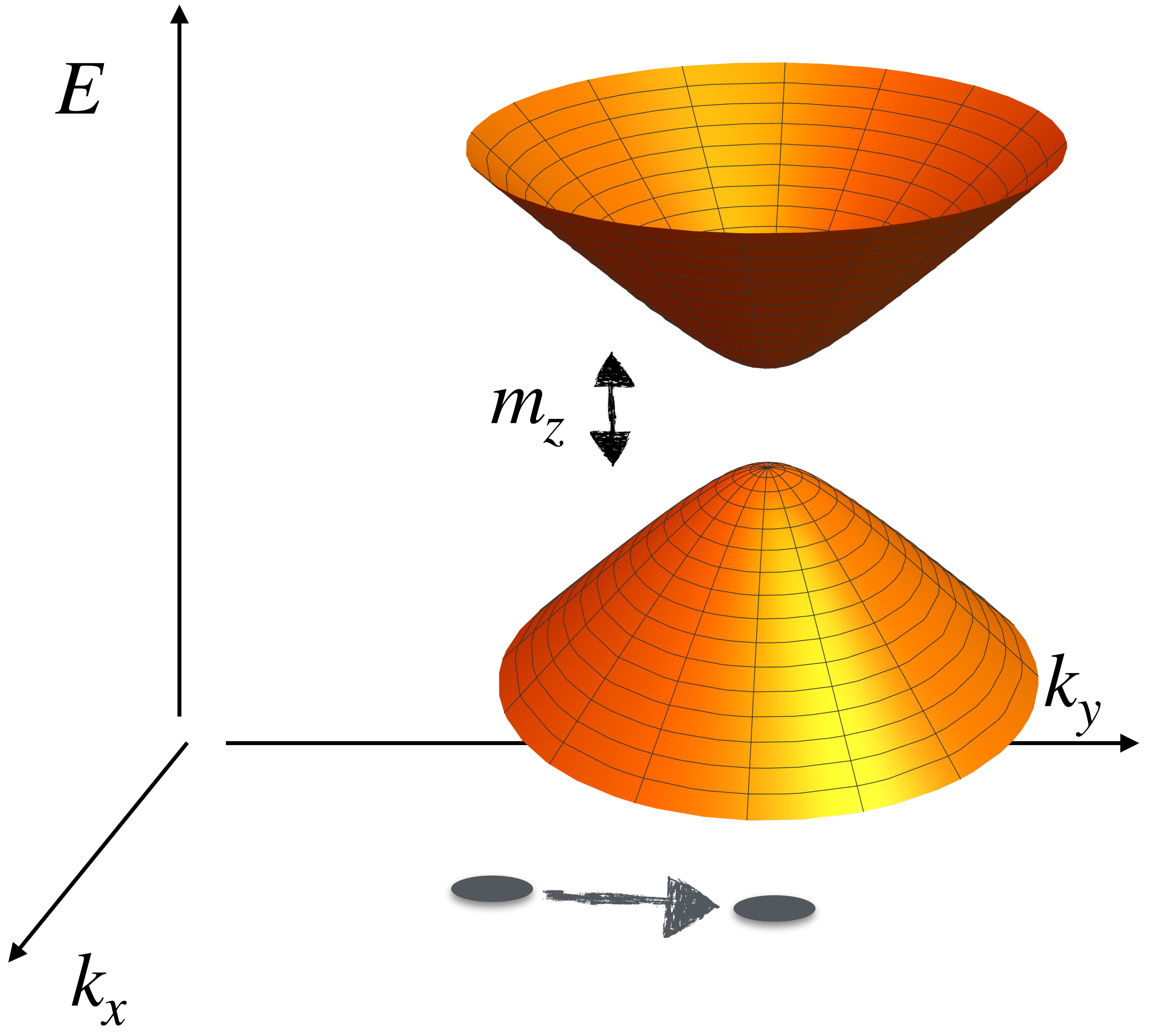


# Half quantized Hall effect at the surface of a topological insulator

- **Dirac electron at the surface:**
  - Bloch Hamiltonian ($v_F \simeq 5\ 10^5$ m/s):

$$H_{\text{surface}} = \hbar v_F \left( k_y \sigma_x - k_x \sigma_y \right) = \hbar v_F (\mathbf{k} \times \mathbf{e}_z) \,.\, \sigma$$

  - Energies: $E_{\text{surface}} = \pm \sqrt{(\hbar v_F k_x)^2 + (\hbar v_F k_y)^2} = \hbar v_F |\mathbf{k}|$




- **Magnetic impurities (e.g. Cr in (Bi,Sb)2Te3):**
  - $H_{imp} = J \sum_{imp} \mathbf{S_i} \,.\, \sigma = \mathbf{m} \,.\, \sigma$
  - Opens a gap: $E_{\text{surface}} = \pm \sqrt{(\hbar v_F k_x + m_y)^2 + (\hbar v_F k_y - m_x)^2 + m_z^2}$
  - At low E, $H_{\text{eff}} = h_x \sigma_x + h_y \sigma_y + h_z \sigma_z$ with $\mathbf{h}(k_x, k_y) = (\hbar v_F k_y, -\hbar v_F k_x, m_z)$

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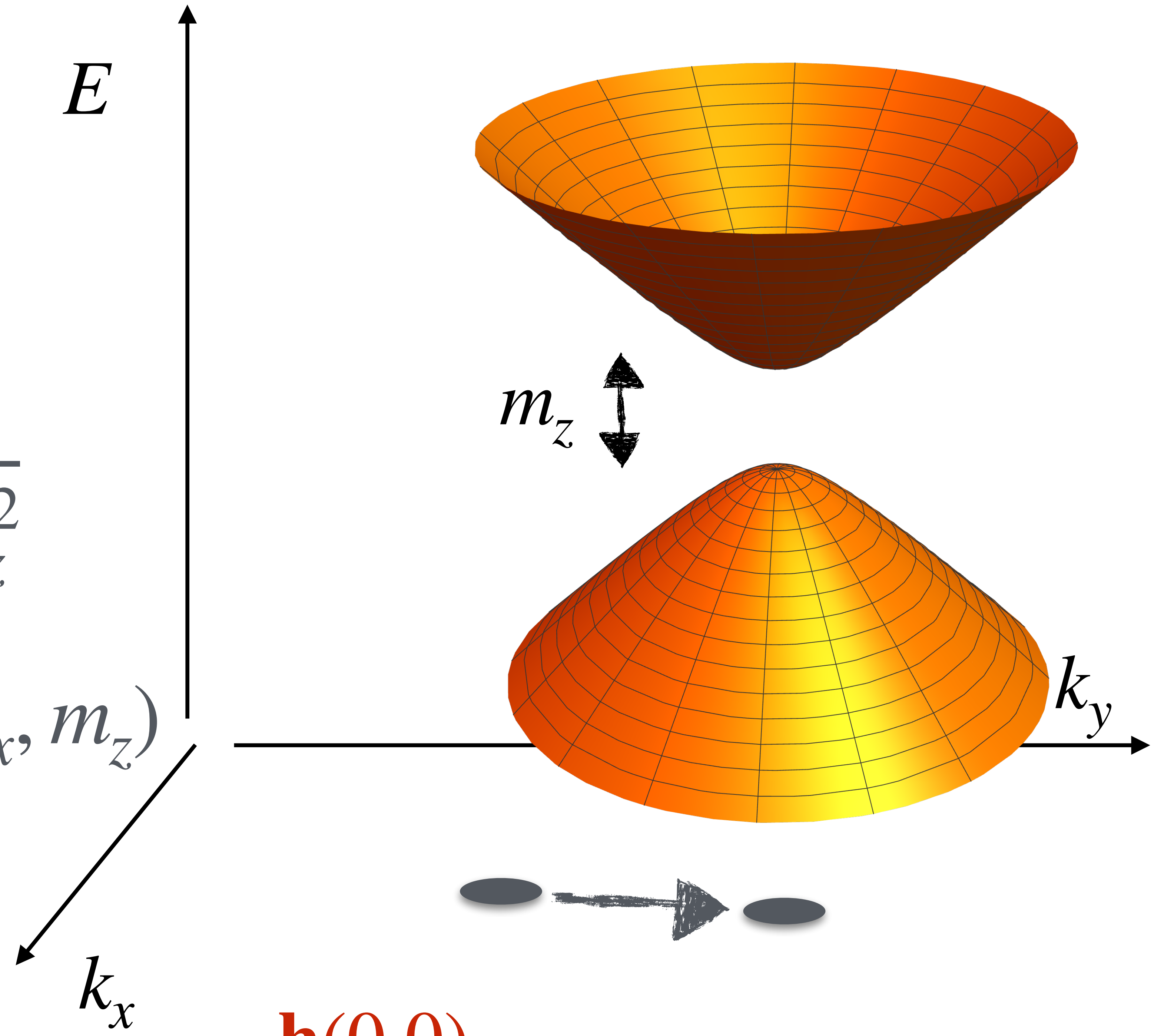
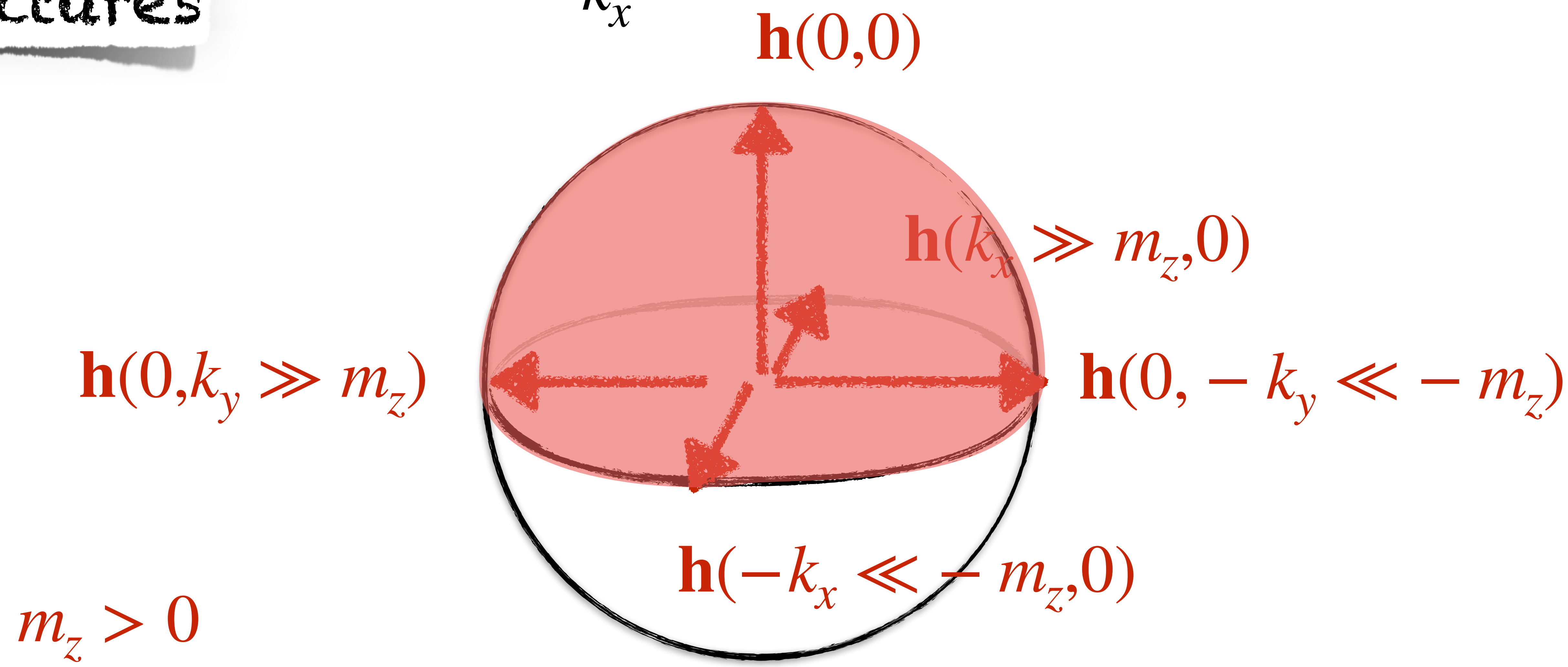



Adolfo's lectures

- **Hall conductivity:**
  - $\sigma_{xy} = -\dfrac{e^2}{h}\dfrac{1}{4\pi}\displaystyle\int dk_x dk_y\, \mathbf{h} \,.\, \left(\partial_{k_x}\mathbf{h} \times \partial_{k_y}\mathbf{h}\right)$

    $= -\dfrac{\text{sgn}(m_z)}{2}\dfrac{e^2}{h}$



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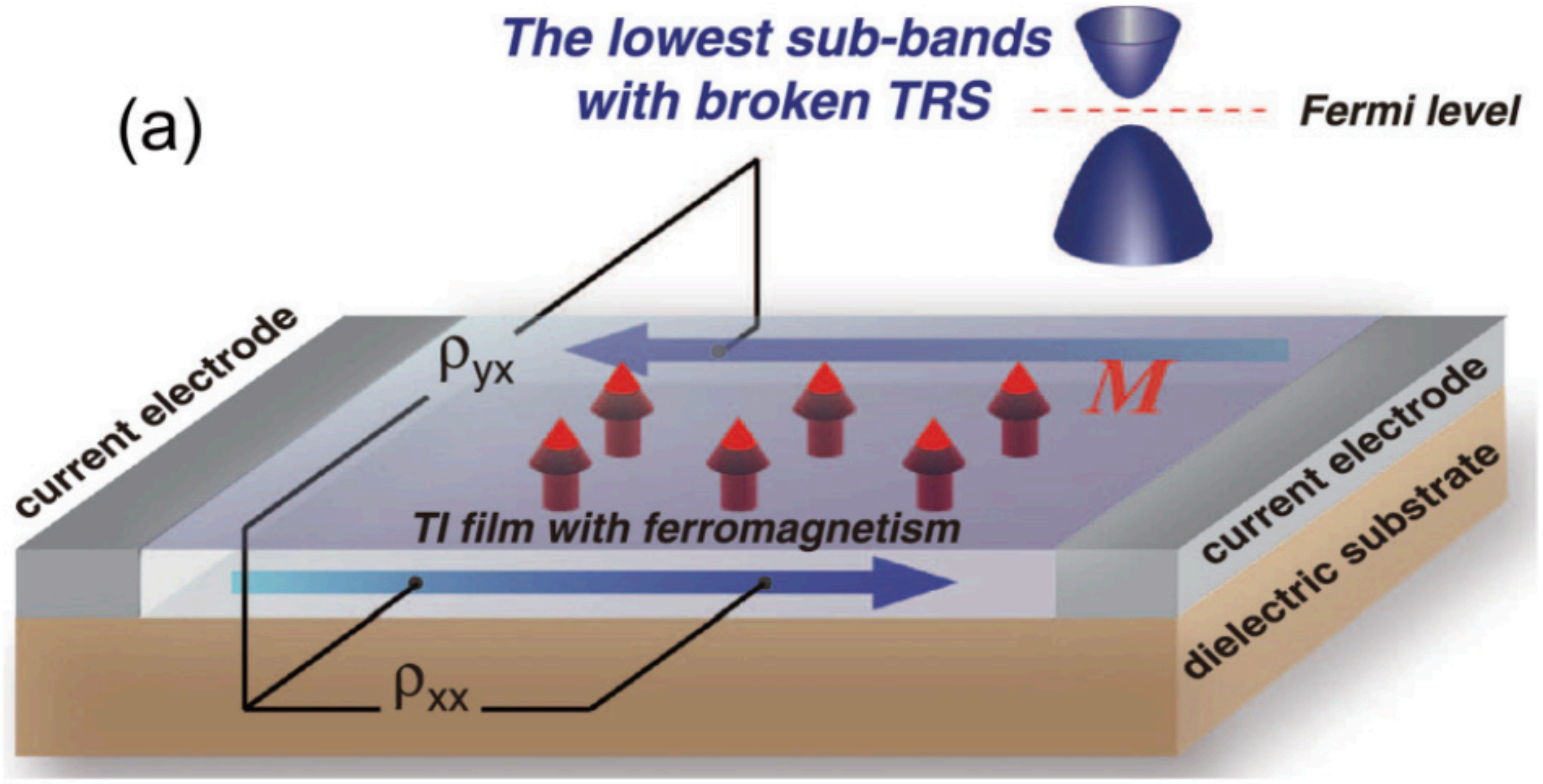
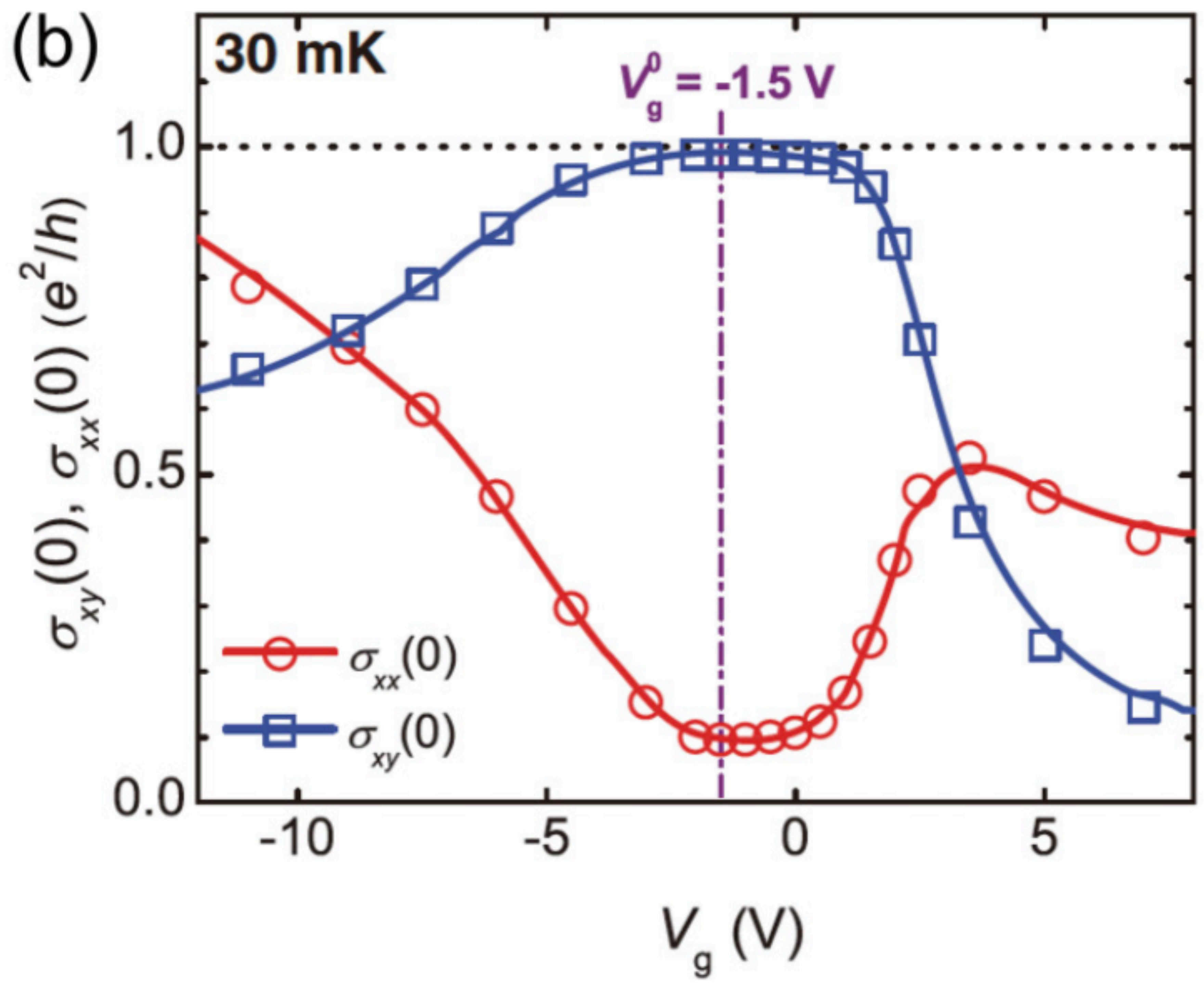
Chang et al., Science (2013)
A. Sekine, K. Nomura, J. Appl. Phys. (2021)

- **Hall conductivity:**

  - $\sigma_{xy} = -\frac{e^2}{h}\frac{1}{4\pi}\int dk_x dk_y\, \mathbf{h} \cdot \left(\partial_{k_x}\mathbf{h} \times \partial_{k_y}\mathbf{h}\right) = -\frac{\mathrm{sgn}(m_z)}{2}\frac{e^2}{h}$

  - Observed in Cr-doped $(Bi,Sb)_2Te_3$

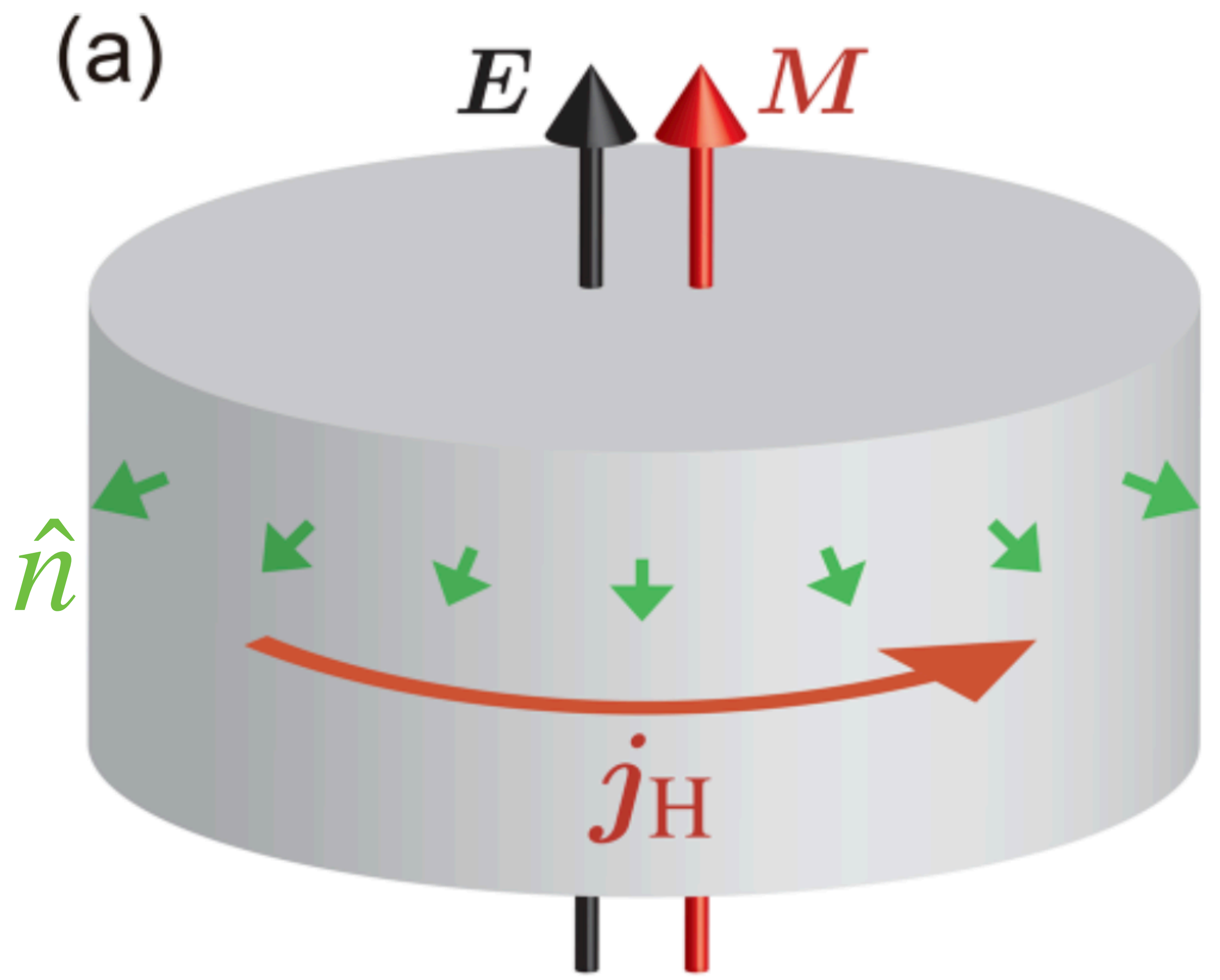


# Phenomenological Magneto-electric effects in topological insulators

- Magnetically doped TI: at the surface, massive Dirac fermions
- Apply $\mathbf{E}$: $\mathbf{j}_H = -\frac{1}{2}\text{sgn}(m)\frac{e^2}{h}\hat{n}\times\mathbf{E}$
- Ampère's law: $|\mathbf{M}| = c^{-1}|\mathbf{j}_H| \Rightarrow \mathbf{M} = \text{sgn}(m)\frac{e^2}{2hc}\mathbf{E}$

An electric field creates a magnetization



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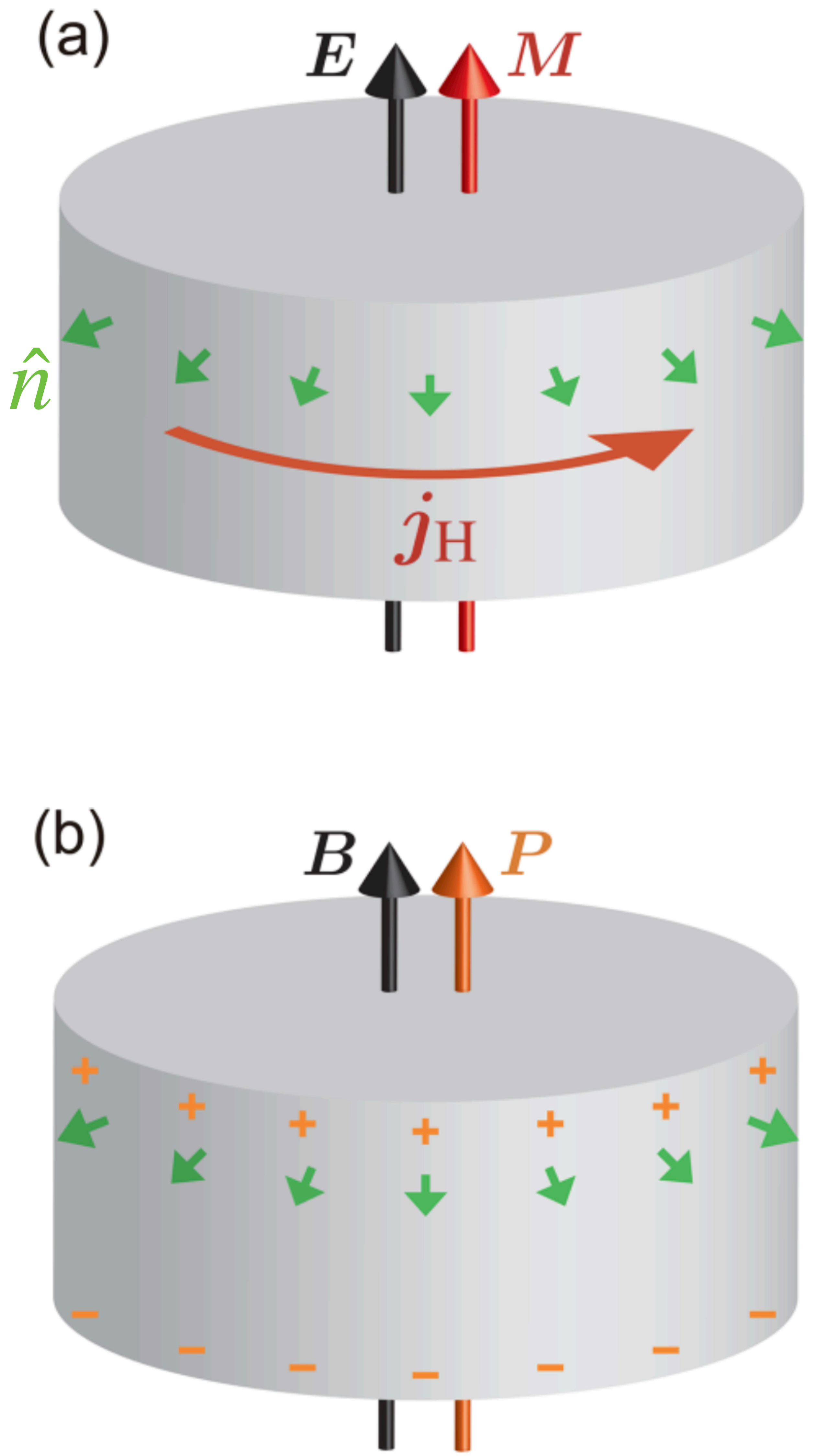
An electric field creates a magnetization

- Apply $\mathbf{B}$, induce $\mathbf{E}^{ind}$ with $\nabla\times\mathbf{E}^{ind} = -\partial_t\mathbf{B}$
- $\mathbf{E}^{ind}$ generates a surface anomalous Hall effect:
  $\mathbf{j}_H = \frac{1}{2}\text{sgn}(m)\frac{e^2}{h}\partial_t\mathbf{B}$
- Using $\mathbf{j}_H = \partial_t\mathbf{P}$ we get $\mathbf{P} = \text{sgn}(m)\frac{e^2}{2hc}\mathbf{B}$

A magnetic field creates a polarization



# Phenomenological Magneto-electric effects in topological insulators

- Magnetically doped TI: at the surface, massive Dirac fermions
- Apply $\mathbf{E}$: $\mathbf{j}_H = -\frac{1}{2}\mathrm{sgn}(m)\frac{e^2}{h}\hat{n}\times\mathbf{E}$
- Ampère's law: $|\mathbf{M}| = c^{-1}|\mathbf{j}_H| \Rightarrow \mathbf{M} = \mathrm{sgn}(m)\frac{e^2}{2hc}\mathbf{E}$

An electric field creates a magnetization

- Apply $\mathbf{B}$, induce $\mathbf{E}^{ind}$ with $\nabla\times\mathbf{E}^{ind} = -\partial_t\mathbf{B}$
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A magnetic field creates a polarization

- **Axion electrodynamics**
  - Modified electrodynamics ($\theta = \pi$)

$$F = -\frac{e^2}{4\pi^2\hbar c}\int d^3\mathbf{r}\ \theta\ \mathbf{E}\,.\,\mathbf{B}$$

  - Magnetization: $\mathbf{M} = -\partial F/\partial\mathbf{B}$

$$\mathbf{M} = \frac{e^2}{4\pi^2\hbar c}\ \theta\ \mathbf{E}$$

  - Electric polarization: $\mathbf{P} = -\partial F/\partial\mathbf{E}$,

$$\mathbf{P} = \frac{e^2}{4\pi^2\hbar c}\ \theta\ \mathbf{B},$$

# Magneto-electric effects in topological insulators

- **Axion electrodynamics**
  - Modified electrodynamics ($\theta = \pi$)

$$F = -\frac{e^2}{4\pi^2\hbar c}\int d^3\mathbf{r}\ \theta\ \mathbf{E}\,.\,\mathbf{B}$$

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$$\mathbf{M} = \frac{e^2}{4\pi^2\hbar c}\ \theta\ \mathbf{E}$$

  - Electric polarization: $\mathbf{P} = -\,\partial F/\partial \mathbf{E}$,

$$\mathbf{P} = \frac{e^2}{4\pi^2\hbar c}\ \theta\ \mathbf{B},$$

- Field theory action:

$$\mathcal{S}_\theta = +\frac{e^2}{4\pi^2\hbar c}\int_{\mathbf{r},t}\ \theta\ \mathbf{E}\,.\,\mathbf{B}$$

$$= \frac{e^2}{32\pi^2\hbar c}\ \theta\ \epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$$

With $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu = (A_0, -\mathbf{A})$

$$\mathbf{E} = -\nabla A_0 - \partial_t \mathbf{A},\quad \mathbf{B} = \nabla \times \mathbf{A}$$

- Topological term: pure surface term ($\theta$ constant)

$$\mathcal{S} = \frac{e^2}{8\pi^2\hbar c}\ \theta\int_{\mathbf{r},t}\epsilon^{\mu\nu\rho\lambda}\partial_\mu(A_\nu\partial_\rho A_\lambda)$$

- Magneto-electric effect in the bulk ↔ surface anomalous response

# Magneto-electric effects in topological insulators

Modified (Axion) electrodynamics: $\mathcal{S}_\theta = \dfrac{e^2}{4\pi^2\hbar c}\displaystyle\int_{t,\mathbf{r}} \theta\,\mathbf{E}\,.\,\mathbf{B}$

- **Time Reversal Symmetry**
  - Time Reversal: $\mathbf{E} \to \mathbf{E}, \mathbf{B} \to -\mathbf{B}$
  - Time Reversal symmetry: $\theta = -\theta \pmod{2\pi}$
    - Topological Insulator: $\theta = \pi$
    - Standard Insulator: $\theta = 0$
  - In magnetic insulators (no Time Reversal symmetry): $\theta$ arbitrary, and depend on $\mathbf{r}, t$

- **Inversion Symmetry**
  - Inversion: $\mathbf{E} \to -\mathbf{E}, \mathbf{B} \to +\mathbf{B}$
  - Inversion symmetry: $\theta = -\theta \pmod{2\pi}$
    - Topological Insulator: $\theta = \pi$
    - Standard Insulator: $\theta = 0$

# Magneto-electric effects in topological insulators

## Axion electrodynamics

- Modified electrodynamics:

$$\mathcal{S}_\theta = \frac{e^2}{4\pi^2\hbar c}\int_{t,\mathbf{r}} \theta\, \mathbf{E}\,.\,\mathbf{B}$$

- Electric polarization: $\mathbf{P} = -\,\partial F/\partial \mathbf{E}$,

$$\mathbf{P} = \frac{e^2}{4\pi^2\hbar c}\,\theta\,\mathbf{B},$$

- Magnetization: $\mathbf{M} = -\,\partial F/\partial \mathbf{B}$

$$\mathbf{M} = \frac{e^2}{4\pi^2\hbar c}\,\theta\,\mathbf{E}$$

## Linear magneto electric coupling

- $$\alpha = \frac{\partial M_i}{\partial E_i} = \frac{\partial P_i}{\partial B_i}$$

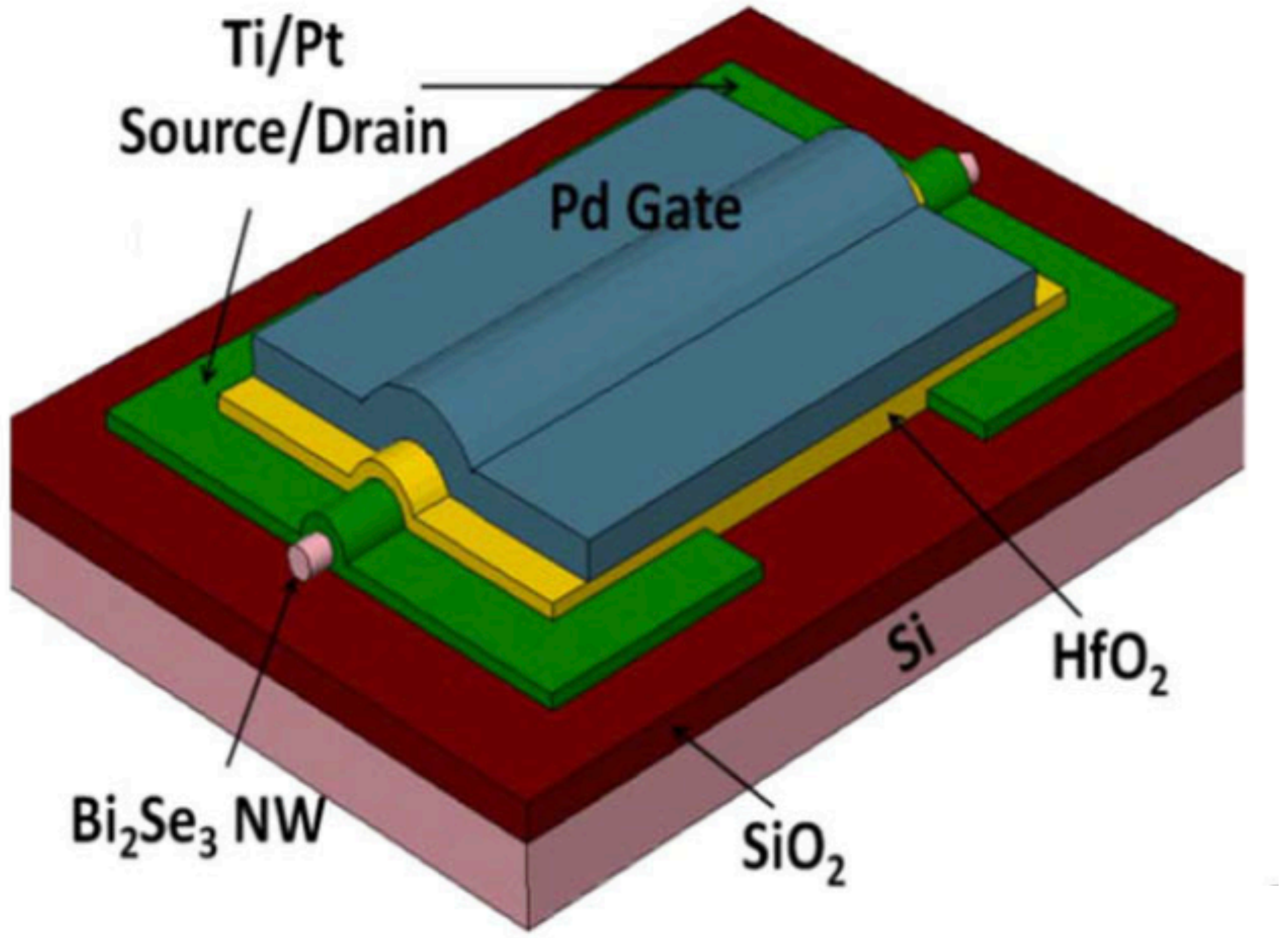
$$= \frac{e^2}{4\pi^2\hbar c}(\theta = \pi)\frac{1}{\mu_0^2 c} \simeq 24.3 \text{ ps/m}$$

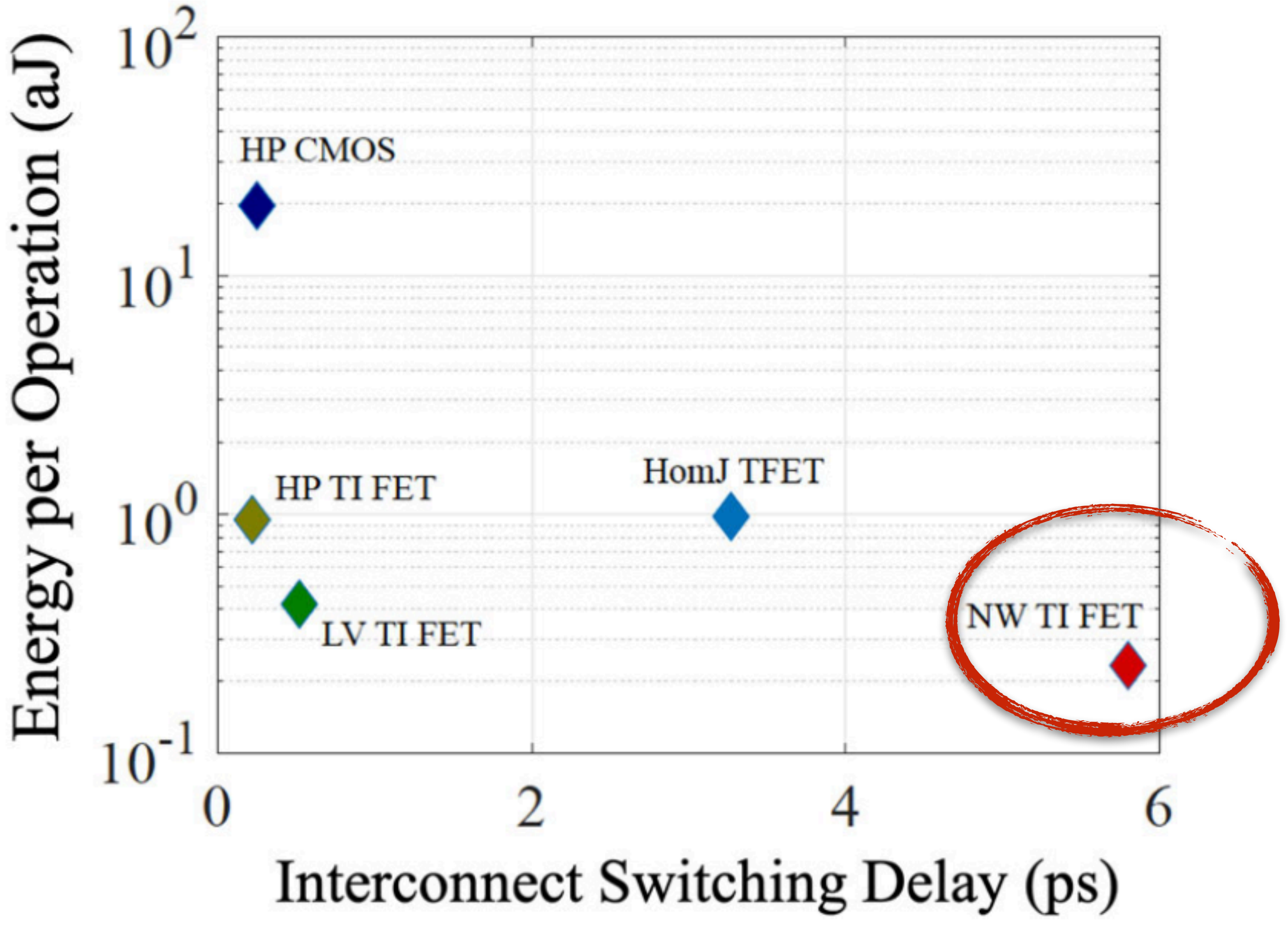
- Magneto-electric (antiferromagnetic) materials ($Cr_2O_3$): $\alpha \simeq 0.7$ ps/mx

# Topological electronics ?

M. Gilbert, Comm. Phys. (2021)
Zhu, H. et al. Sci. Rep.(2013)

- **Topological field-effect transistors**
  - Surface states + bulk states -> reduce the thickness
  - 3.5-nm-thick $Bi_2Se_3$ FET
  - Low energy consumption but slow



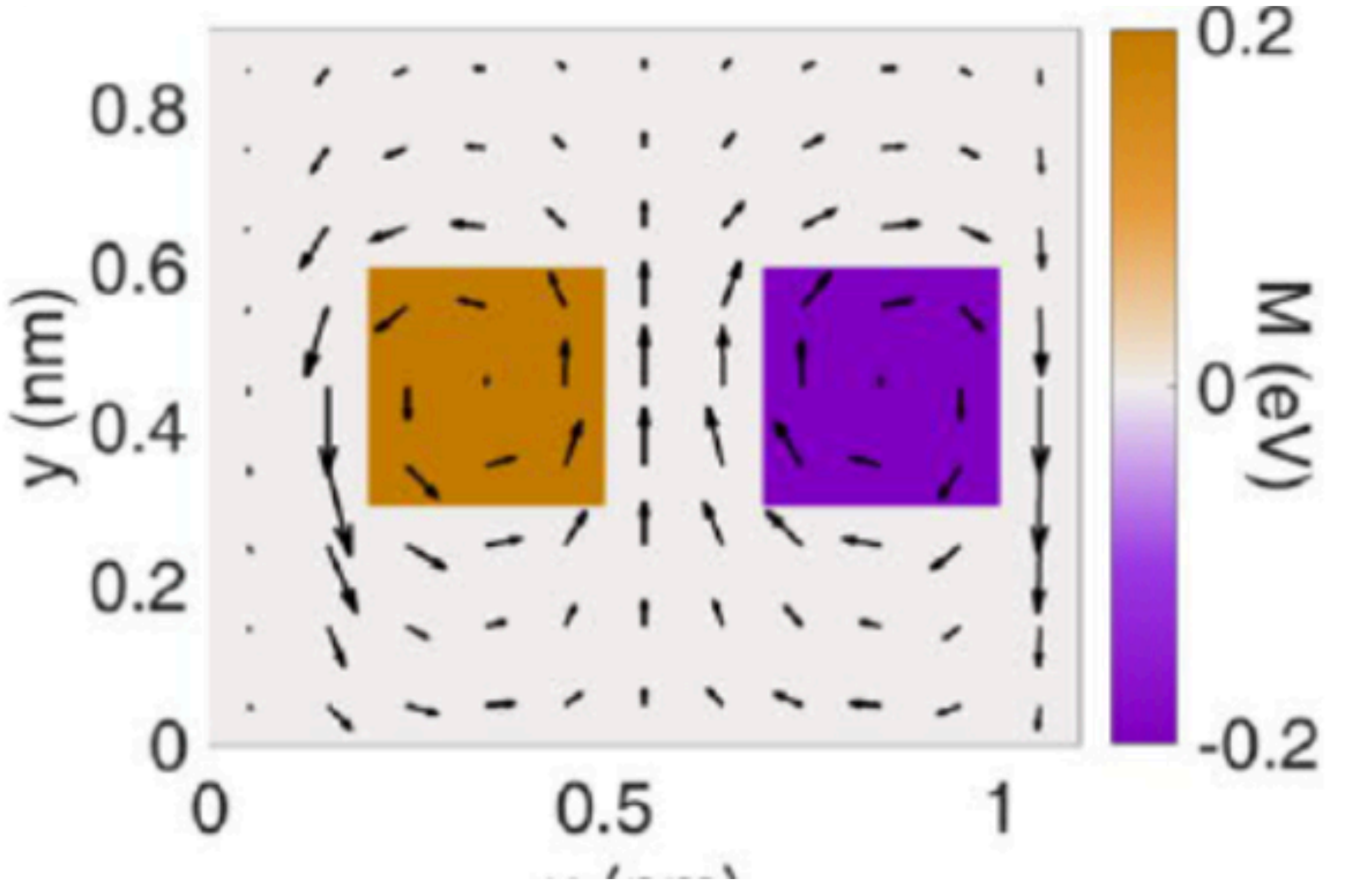



# Topological electronics ?

M. Gilbert, Comm. Phys. (2021)

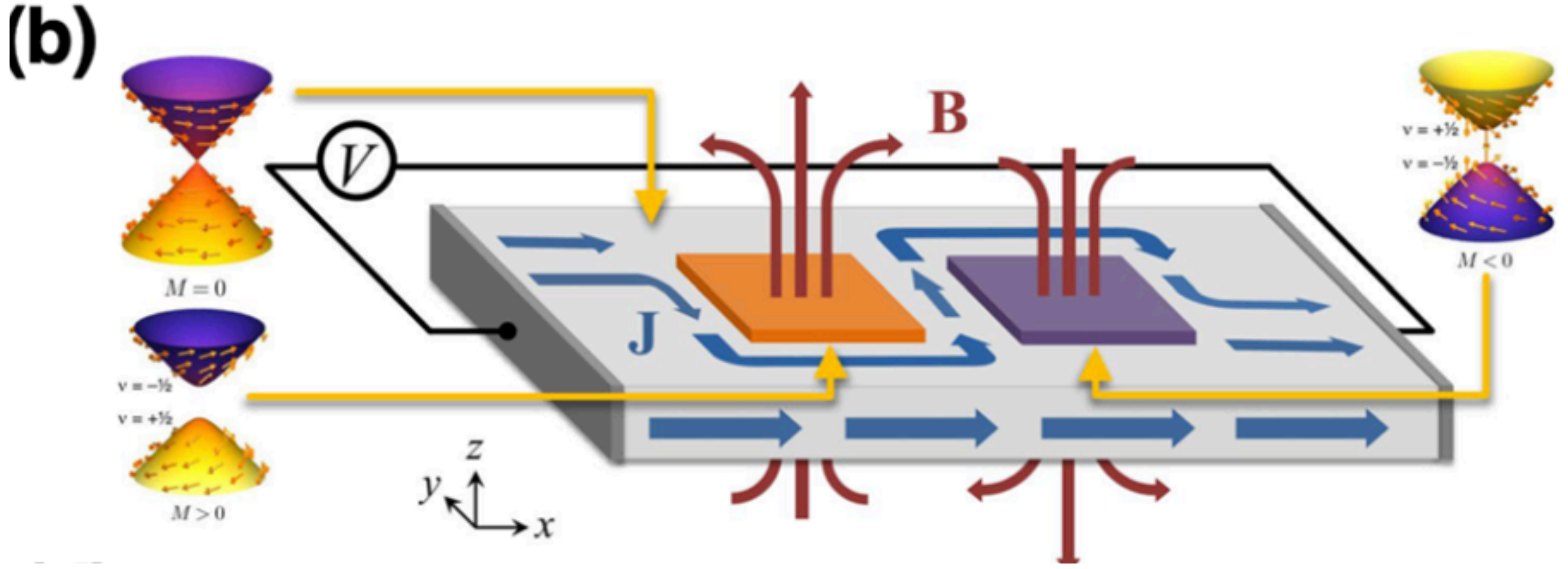
Philip, T. M. & Gilbert, M. Sci. Rep. (2017)

- **Topological magneto-electric effect inductors**
  - high-performance, small-footprint, on-chip inductors
  - Hall effects current around the ferromagnetic islands: concentrate magnetic flux, high inductance




(d)

| Inductor | Cut-off Frequency (GHz) | Inductance (nH/mm²) |
|---|---|---|
| LF Copper[6] | 0.2 | 1700 |
| RF Copper[1] | 6 | 282 |
| CNT[10] | 150 | 23.2 |
| Graphene[7] | 150 | 636 |
| Topological Inductor | 1000 | 930 |



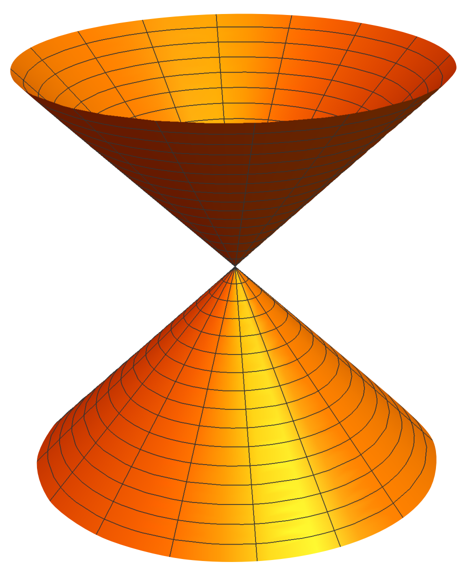
# Outline

- **2D topological insulators**
  - Edge states versus bulk: the Quantum Hall Effect example
  - Modified electrodynamics and quantized Hall response in 2D
- **3D topological insulators**
  - Surface properties
  - Modified electrodynamics in 3D: magneto-electric effects
  - (A choice of some) consequences
- **3D topological semimetals**
  - Recap of topological properties: anomalous Hall effect
  - Modified electrodynamics
  - Non-reciprocity of electrodynamics: consequences on optical and thermal properties

# Weyl, Dirac, and band crossings

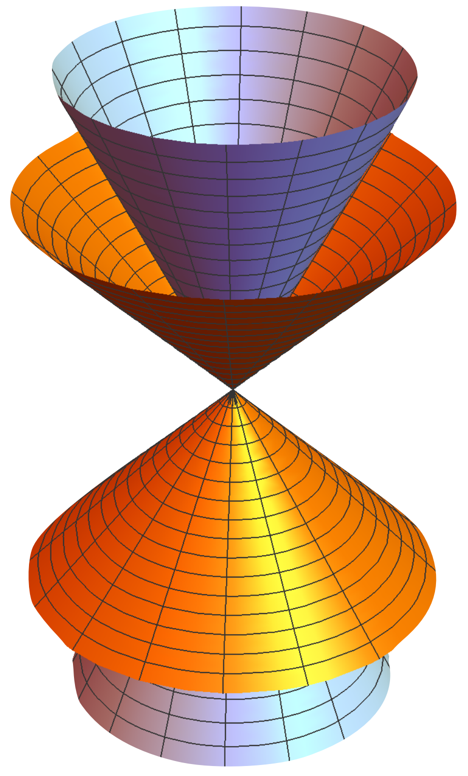
- **Weyl point:** Linear Crossing between **Two Bands** in **D=3**
  - locally Bloch Hamiltonian = Weyl Hamiltonian

    $H(\mathbf{K}+\mathbf{q}) = \chi\hbar v_F\,(q_x\sigma_x + q_y\sigma_y + q_z\sigma_z)$ with a chirality $\chi = \pm 1$
  - come by **pair of opposite chirality**



- **Dirac point:** Linear Crossing between **Four Bands** in **D=3**
  - locally Bloch Hamiltonian = massless Dirac Hamiltonian

$$H(\mathbf{k}=\mathbf{K}+\mathbf{q}) = \begin{pmatrix} H_{\mathrm{Weyl}}(\chi=+1,\mathbf{q}) & 0 \\ 0 & H_{\mathrm{Weyl}}(\chi=-1,\mathbf{q}) \end{pmatrix}$$

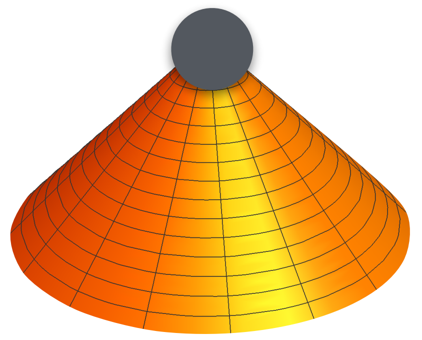


… Old subject revisited recently

W. C. Herring, *Accidental Degeneracy in the Energy Bands of Crystals* (1937)

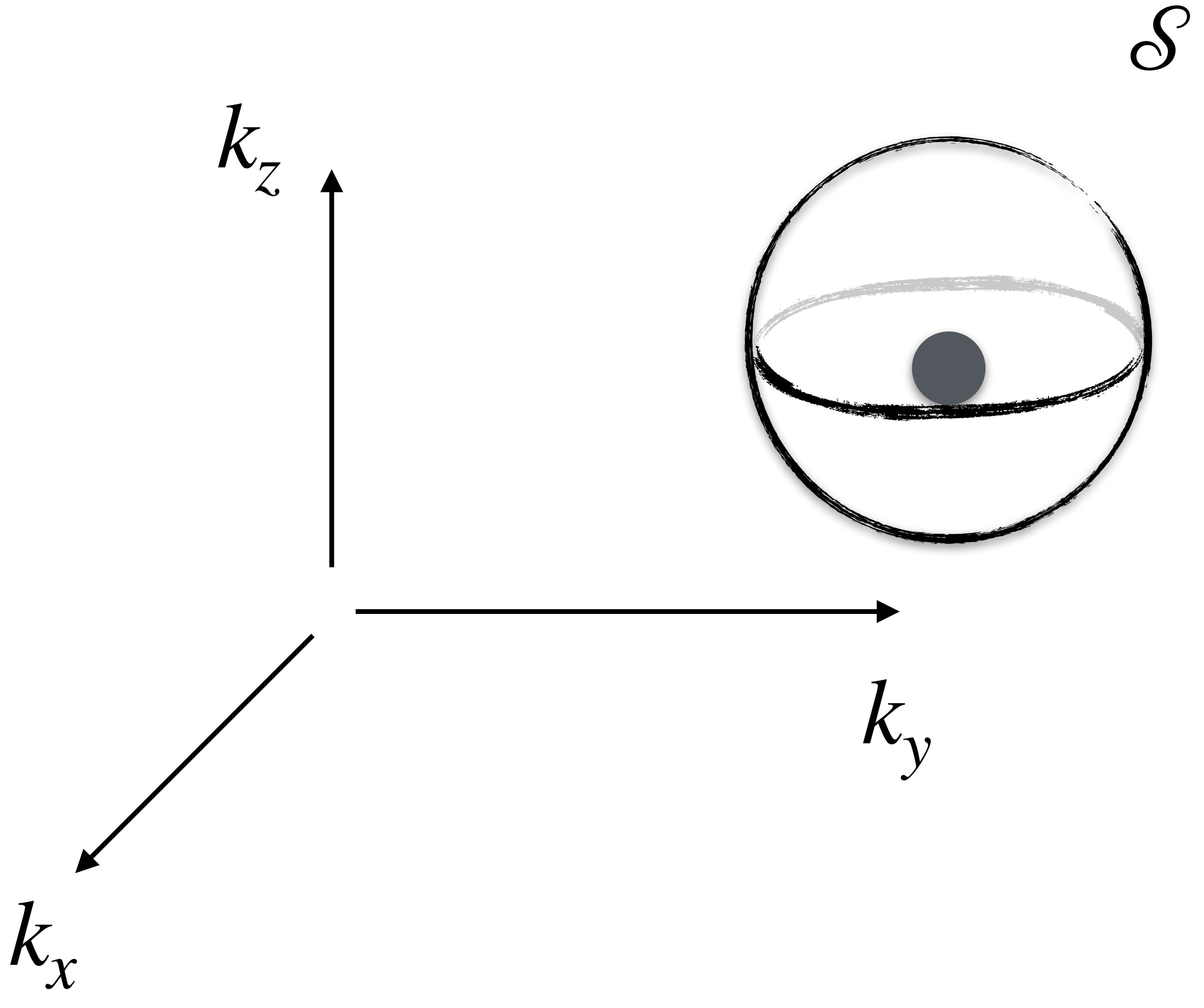
# Topological Properties of a Weyl point

- **Weyl point:** Linear Crossing between **Two Bands** in **D=3**
  - locally Bloch Hamiltonian = Weyl Hamiltonian
    $H(\mathbf{K}+\mathbf{q}) = \chi\hbar v_F\,(q_x\sigma_x + q_y\sigma_y + q_z\sigma_z)$ with a chirality $\chi = \pm 1$




  - come by **pair of opposite chirality**

  - Focus on states $\psi\_(\mathbf{k})$ below the crossing
  - Chern number around the Weyl point: $n\_(\mathbf{K}) = \frac{1}{2\pi}\oint_{\mathcal{S}} F\_(\mathbf{k}) = \chi$ (Berry Curvature on)
  - Weyl point = Berry monopole



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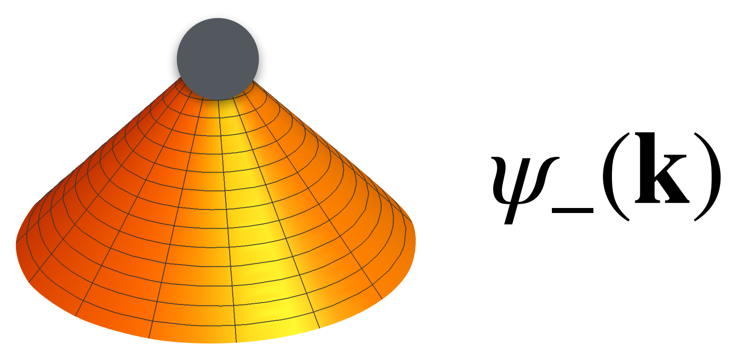
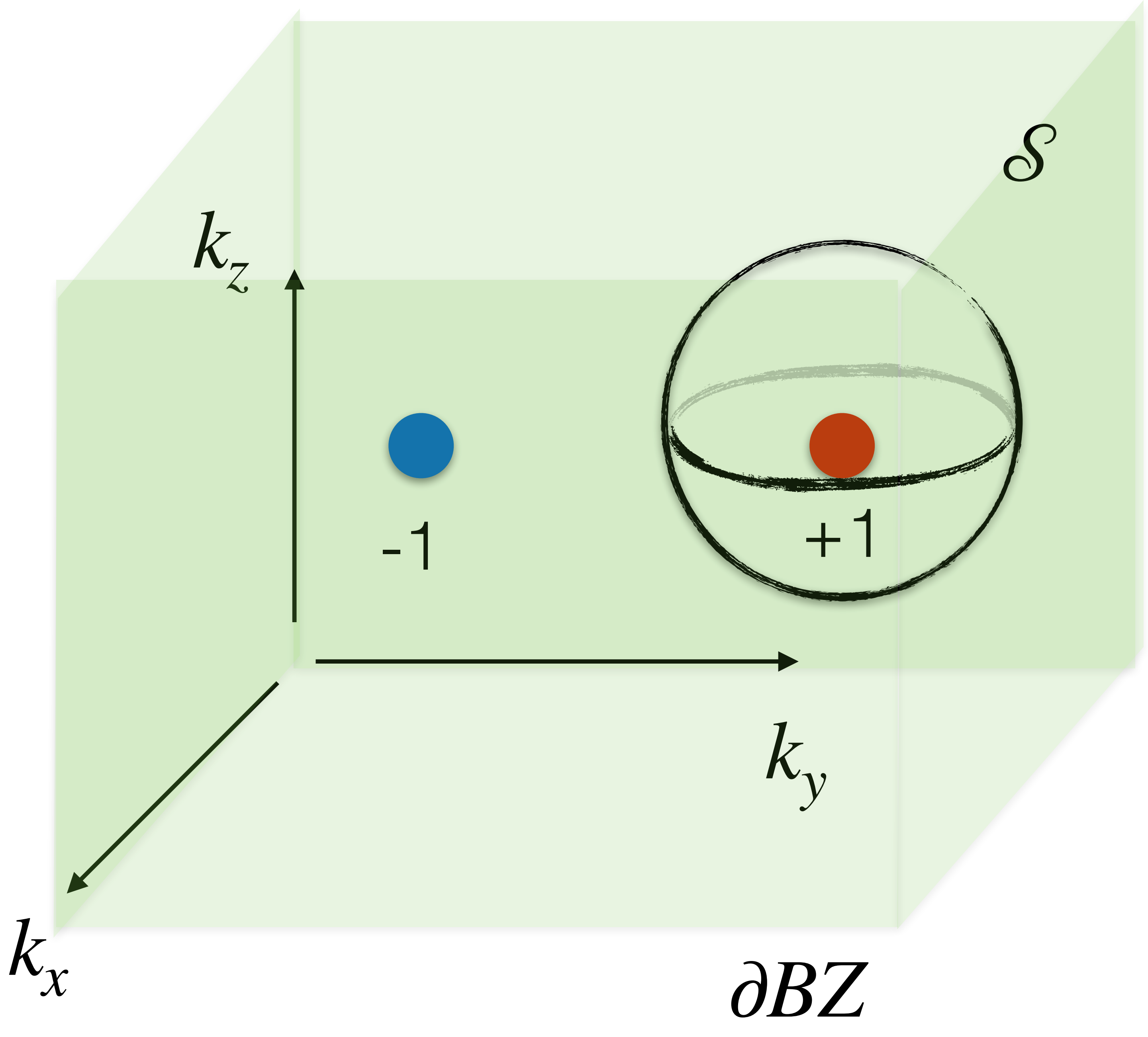
  - Chern number around the Weyl point: $n\_(\mathbf{K}) = \frac{1}{2\pi}\oint_{\mathcal{S}} F\_(\mathbf{k}) = \chi$ (Berry Curvature on $\mathcal{S}$)
  - Weyl point = Berry monopole (analog of Dirac monopole)
  - $\oint_{\partial BZ} F\_(\mathbf{k}) = 0$
  - Weyl points come by pair $n_ = \pm 1$of **opposite chirality**



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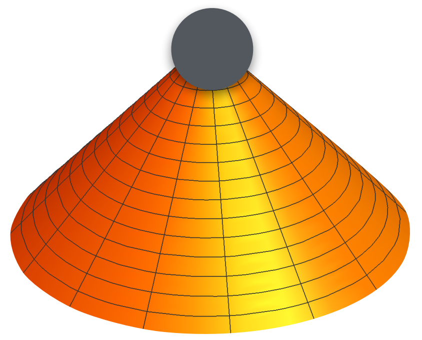
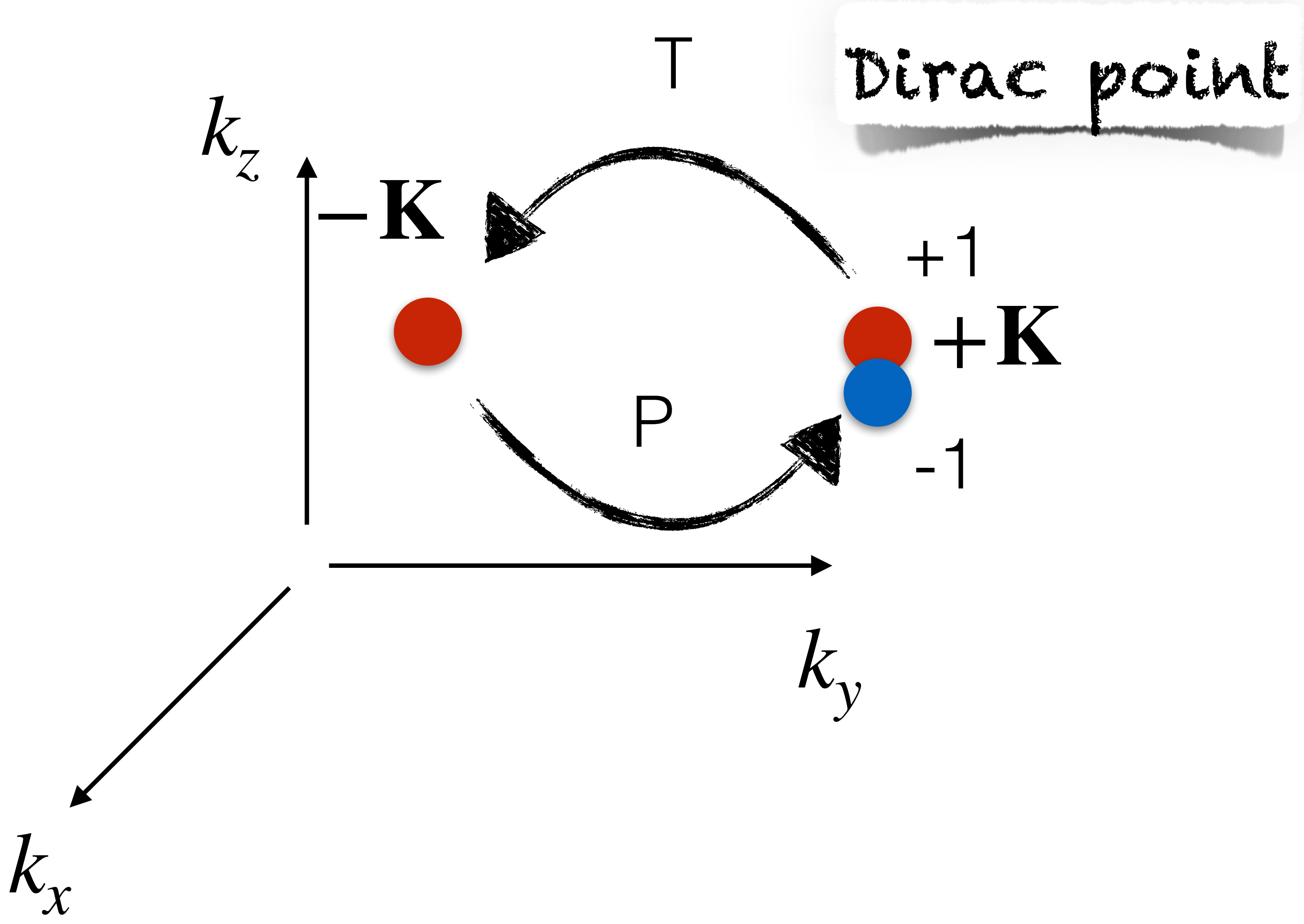
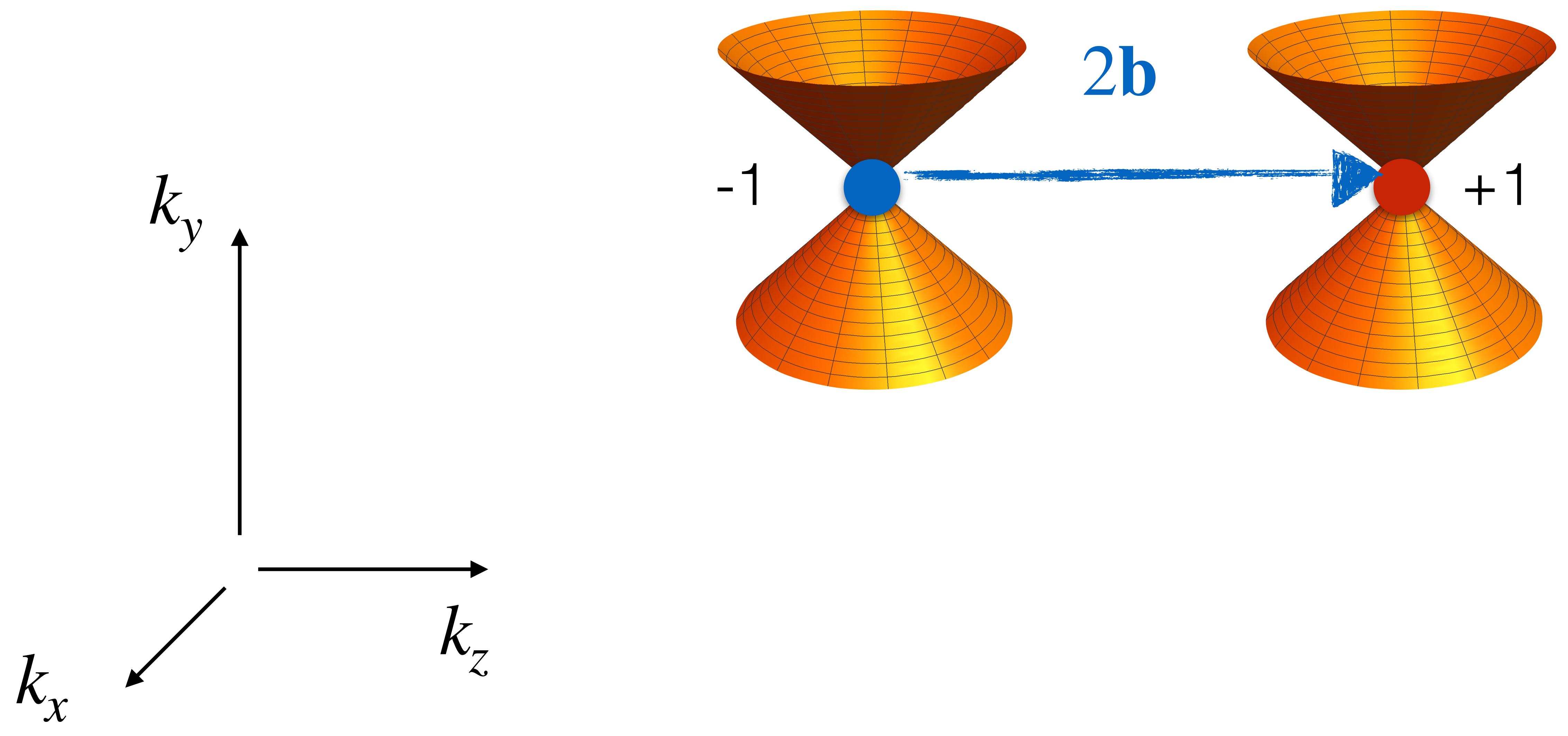
  - come by **pair of opposite chirality**
  - Chern number around the Weyl point: $n\_(\mathbf{K}) = \frac{1}{2\pi}\oint_{\mathcal{S}} F\_(\mathbf{k}) = \chi$
  - Time Reversal Symmetry : $n\_(\mathbf{K}) = +\,n\_(-\mathbf{K})$
  - Parity: $n\_(\mathbf{K}) = -\,n\_(-\mathbf{K})$
  - Weyl point: break either P or T
  - P breaking Weyl semimetal: at least 4 Weyl points
  - T breaking Weyl semimetal: at least 2 Weyl points



# Fermi Arcs and Hall effect of Weyl Semi-Metals

[Wan *et al*, (2011), Hosur *et al*, 2013]




Brillouin Zone (Bulk)

**2 Weyl points:** Linear Crossing between **Two Bands** in **D=3**

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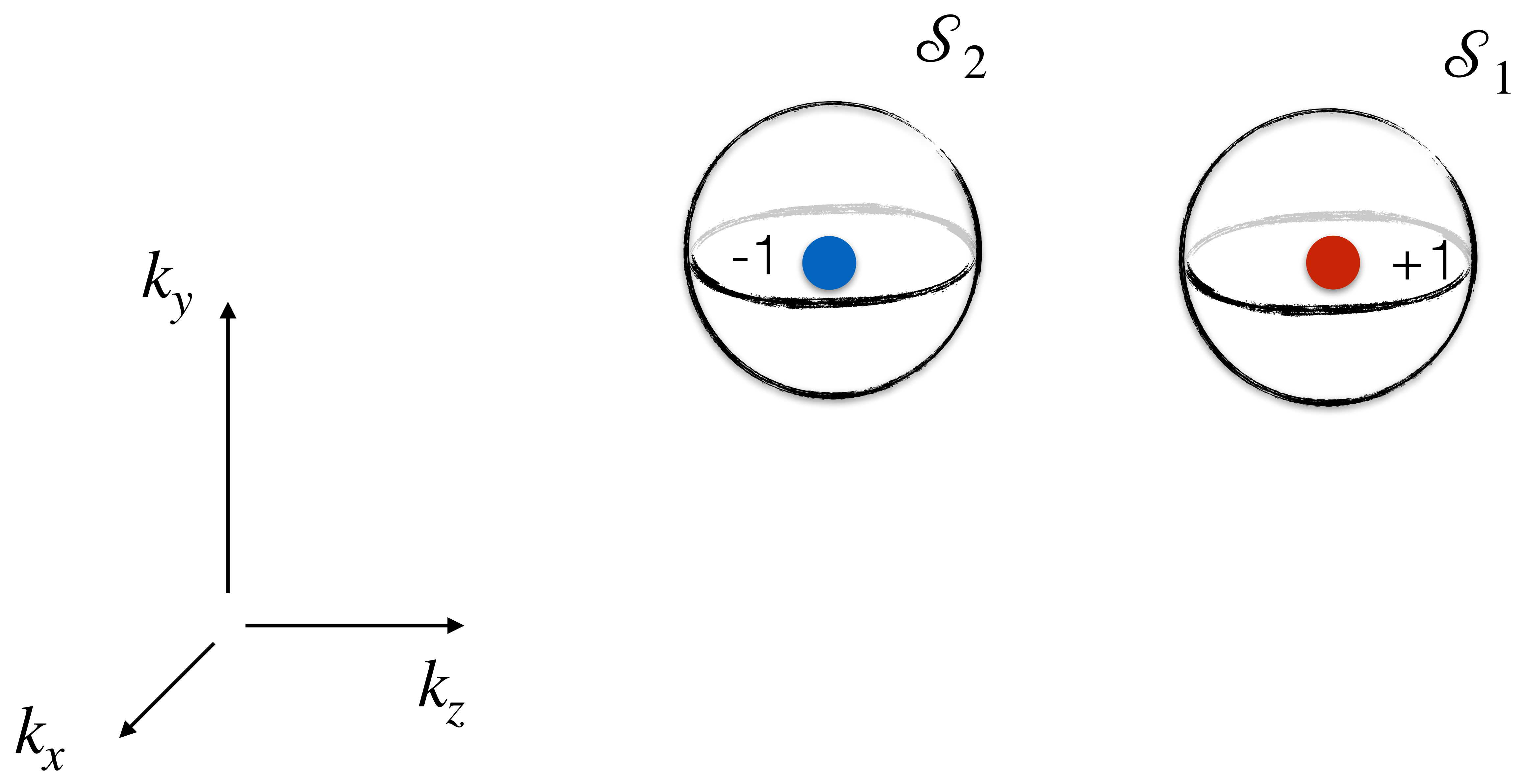


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  $$n\_(1/2) = \frac{1}{2\pi}\oint_{\mathcal{S}} F\_(\mathbf{k}) = \pm 1$$

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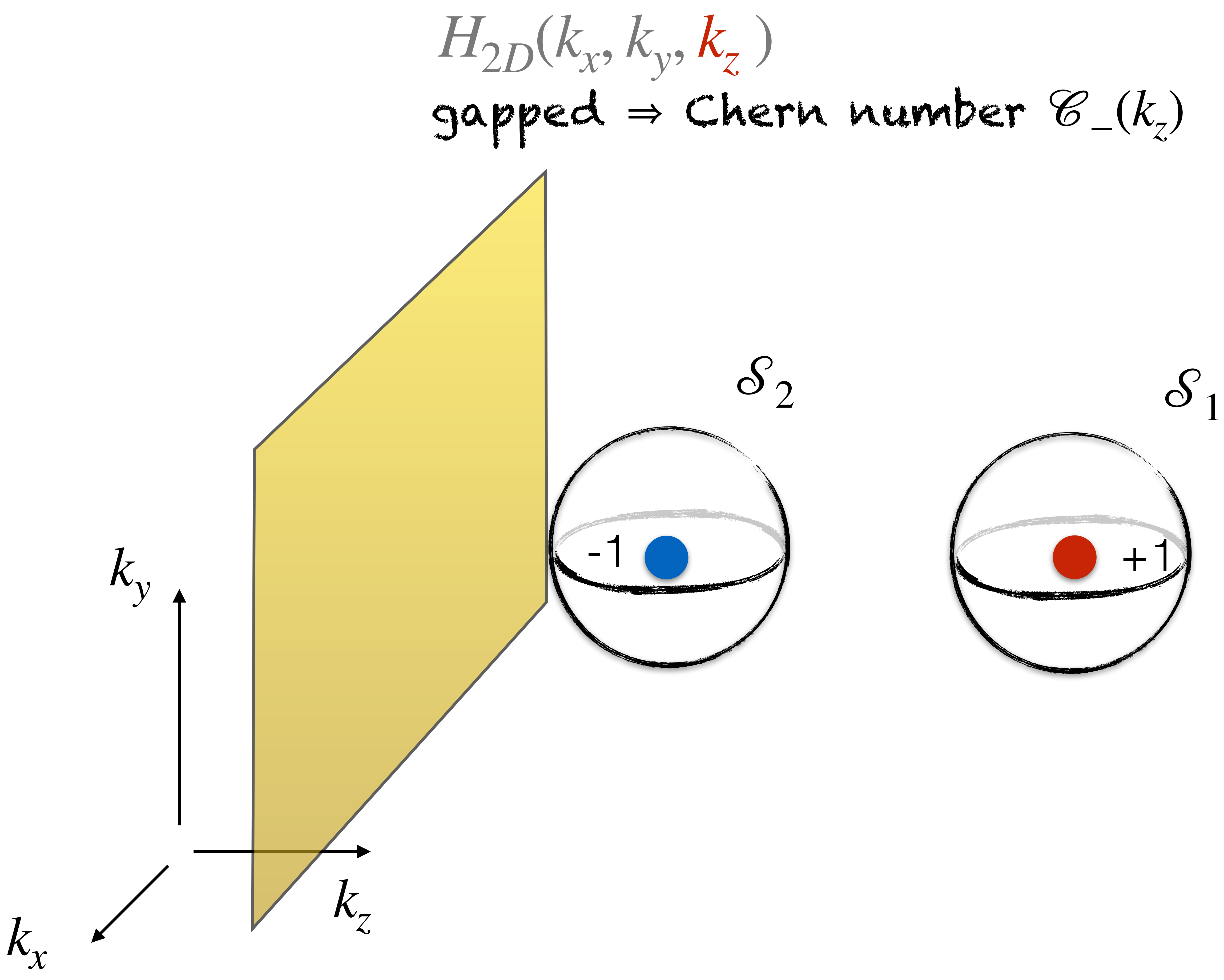



Brillouin Zone (Bulk)

- Chern number around the Weyl points:

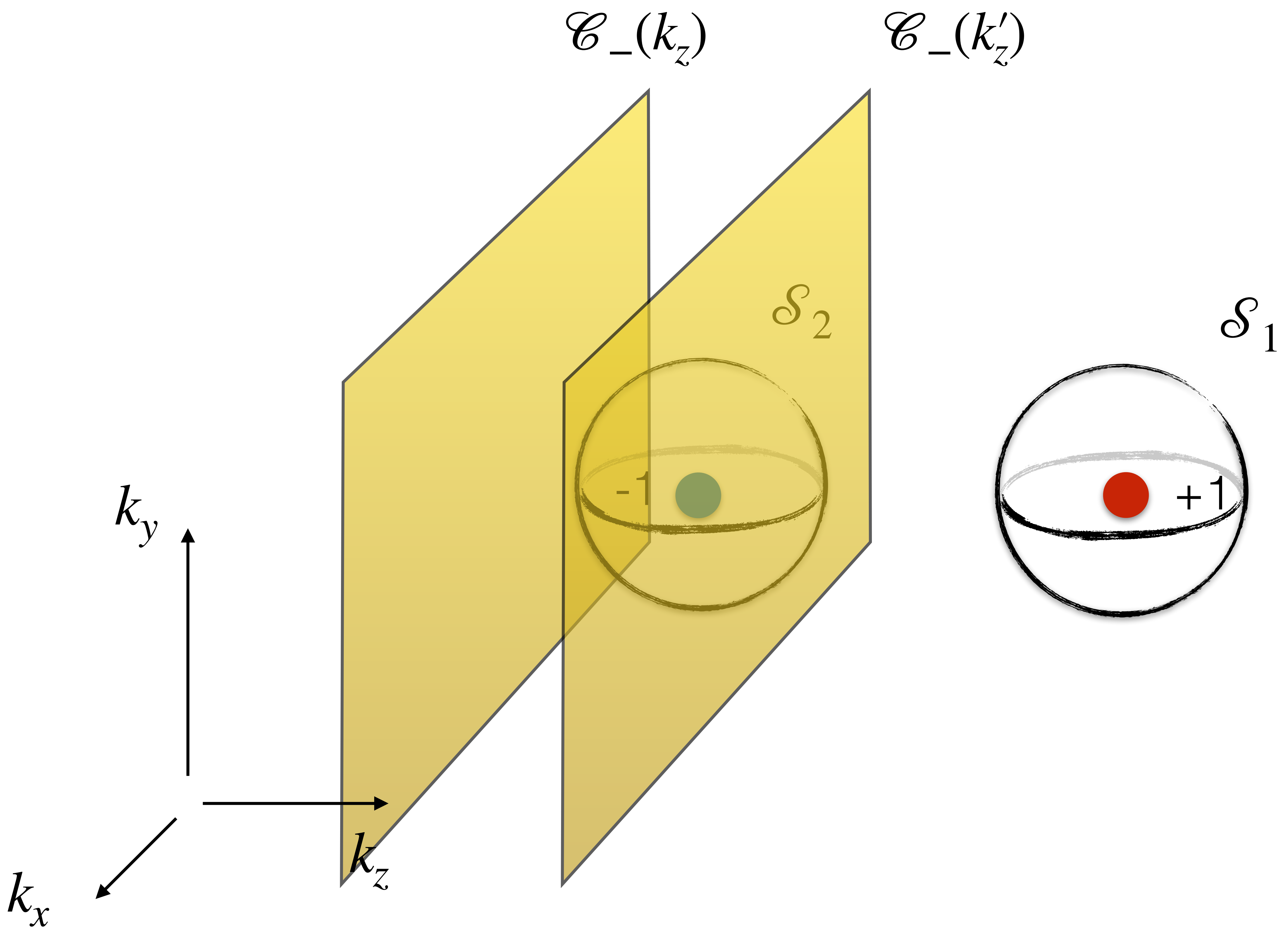
$$n\_(1/2) = \frac{1}{2\pi}\oint_{\mathscr{S}} F\_(\mathbf{k}) = \pm 1$$

- Consider 2D Bloch Hamiltonian $H_{2D}(k_x, k_y, k_z)$ at fixed $k_z$ : 2 bands with a gap

- $$\mathscr{C}\_(k_z) = \frac{1}{2\pi}\oint_{k_x,k_y} F\_(\mathbf{k}) \in \mathbb{Z}$$

# Fermi Arcs and Hall effect of Weyl Semi-Metals

[Wan *et al*, (2011), Hosur *et al*, 2013]

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- Difference $\mathscr{C}\_(k_z') - \mathscr{C}\_(k_z') = n\_(2)$

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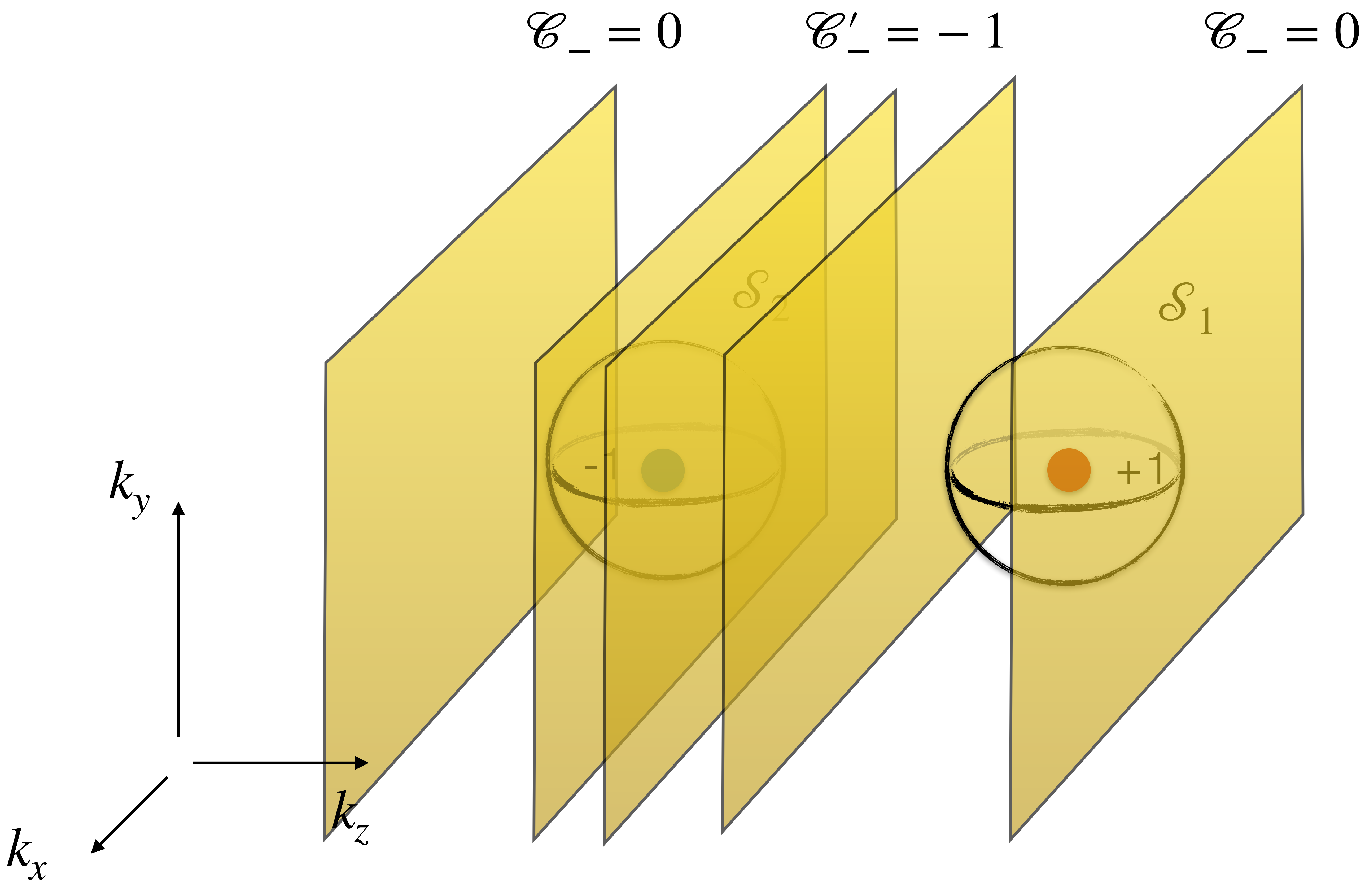
[Wan *et al*, (2011), Hosur *et al*, 2013]




Brillouin Zone (Bulk)

- Chern number around the Weyl points:

$$n_-(1/2) = \frac{1}{2\pi}\oint_{\mathscr{S}} F_-(\mathbf{k}) = \pm 1$$

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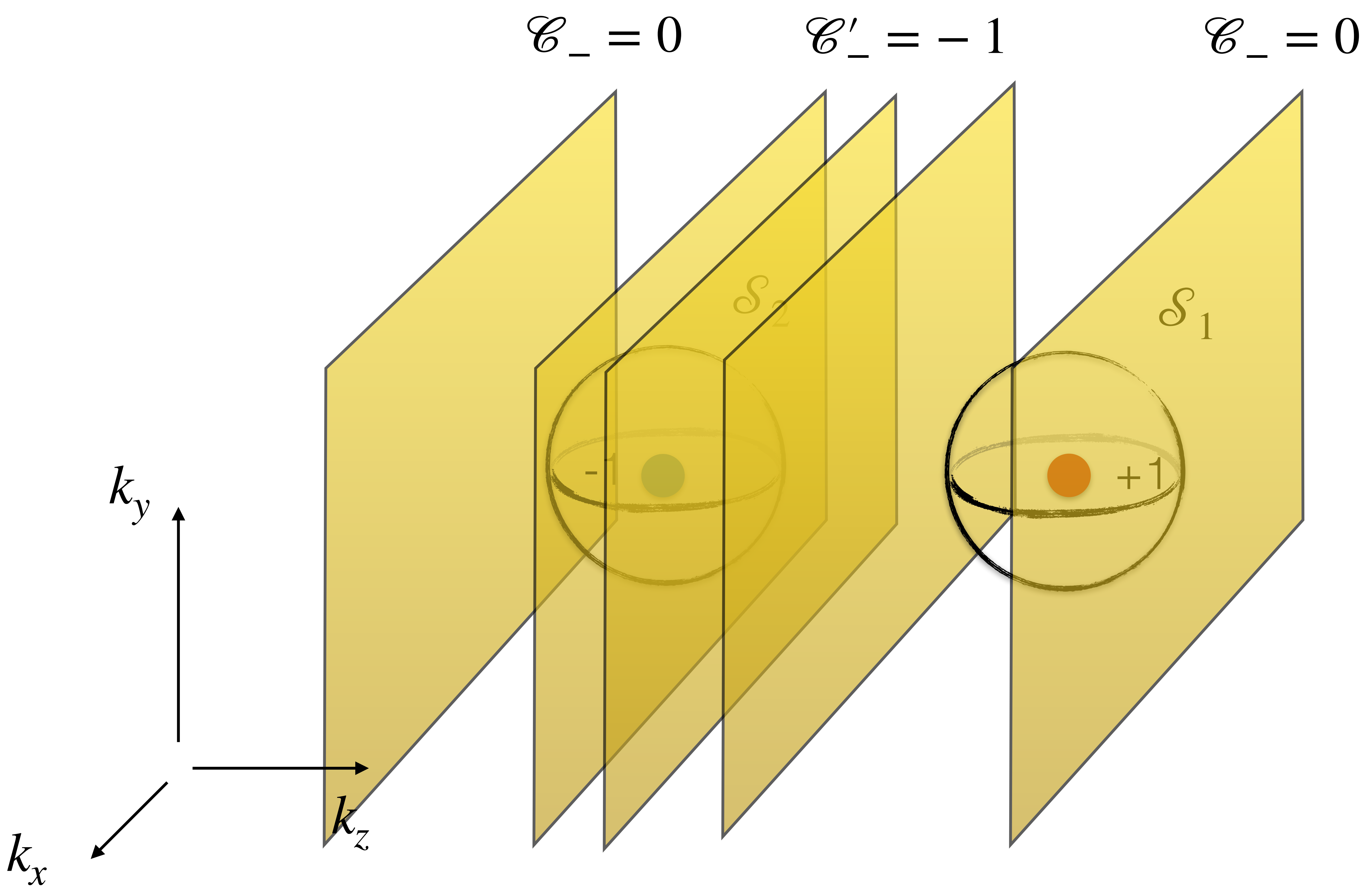
- $\mathscr{C}_-(k_z) = \frac{1}{2\pi}\oint_{k_x,k_y} F_-(\mathbf{k}) \in \mathbb{Z}$
- Difference $\mathscr{C}_-(k_z') - \mathscr{C}_-(k_z') = n_-(2)$
- $\mathscr{C}'_- - \mathscr{C}_- = -1$
- $\mathscr{C}_- - \mathscr{C}'_- = 1$

# Fermi Arcs and Hall effect of Weyl Semi-Metals

[Wan *et al*, (2011), Hosur *et al*, 2013]




Brillouin Zone (Bulk)

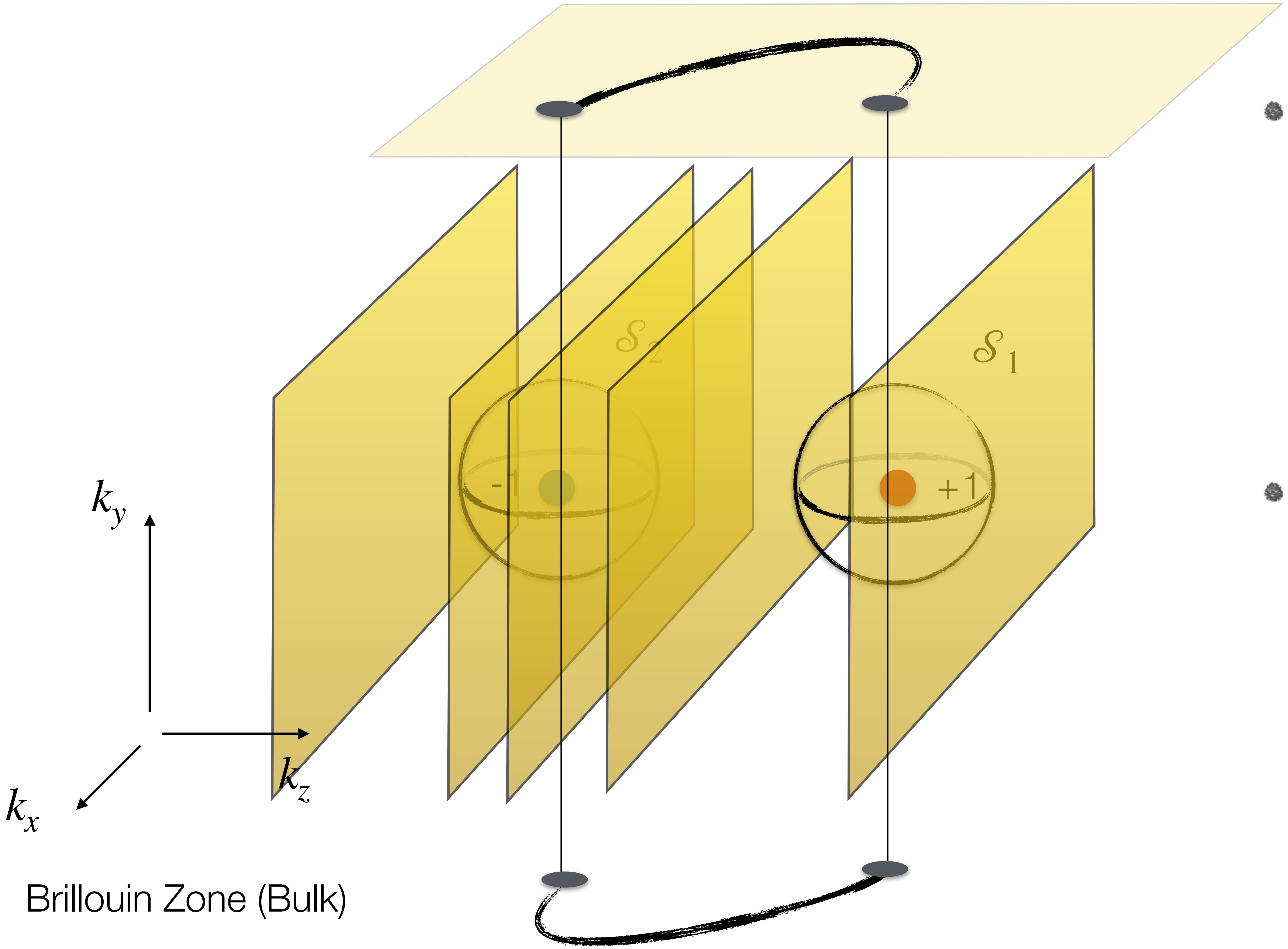
- Hall effect $\perp \mathbf{b}$

$$\sigma_{xy} = -\left(\#k_z \in [\mathbf{K}, \mathbf{K}']\right)\frac{e^2}{h}$$

$$= -\frac{2b}{2\pi/L}\frac{e^2}{h}$$

# Fermi Arcs and Hall effect of Weyl Semi-Metals

[Wan *et al*, (2011), Hosur *et al*, 2013]

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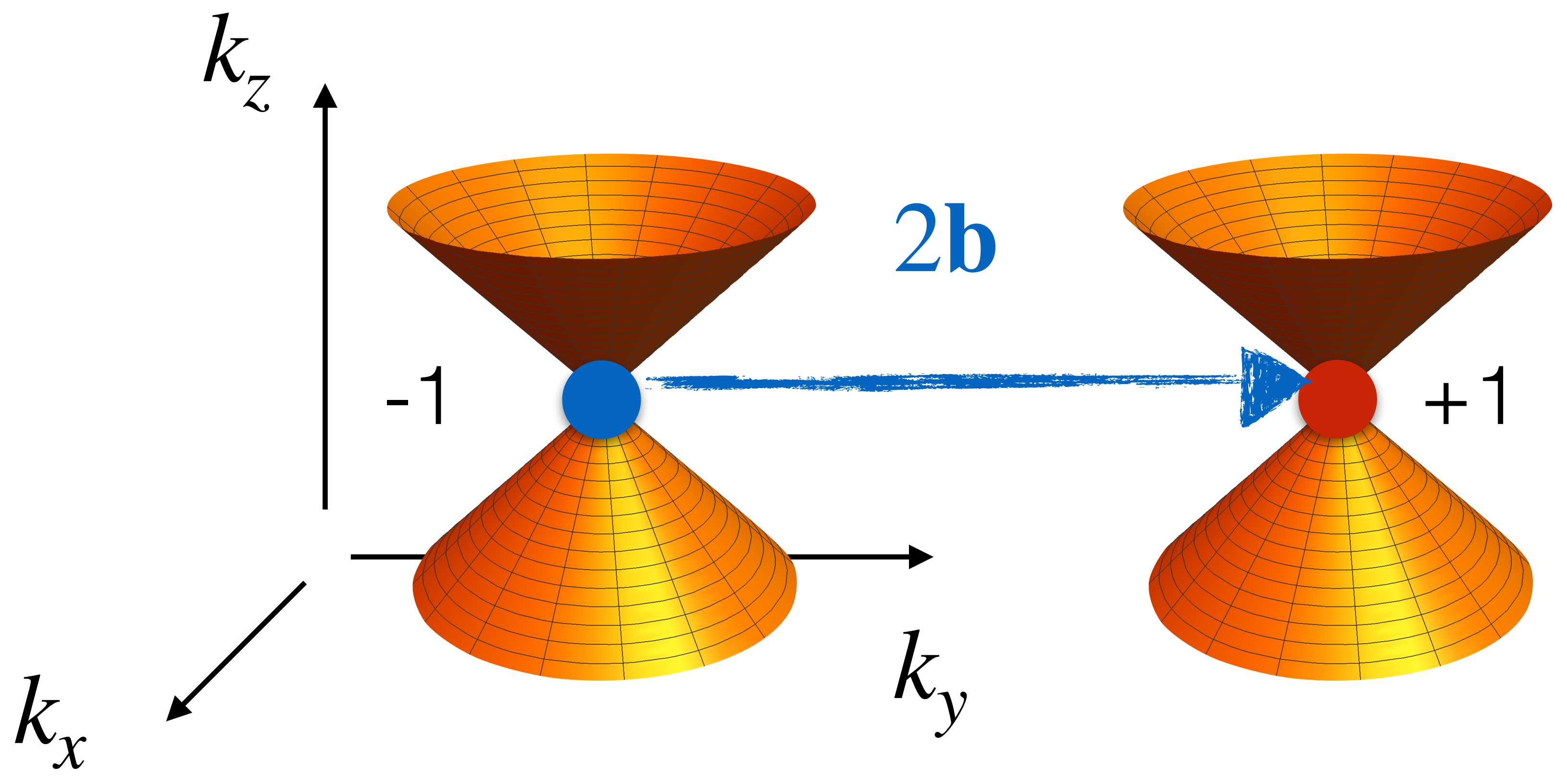
$$= -\frac{2b}{2\pi/L}\frac{e^2}{h}$$

- Fermi arcs of surface states

# Magneto-electric effects in Weyl semi-metals

## Electrodynamics of an insulator

- Standard Maxwell Lagrangian (isotropic):
  $$\mathscr{L}_0 = \frac{\epsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2 - \rho\phi + \mathbf{j}\,.\,\mathbf{A}$$
- Action $\mathscr{S} = \int d^2\mathbf{r}dt\ \mathscr{L}$
- $\frac{\delta\mathscr{S}}{\delta\phi} = 0 = -\rho + \epsilon_0\ \nabla\,.\,\mathbf{E}$
- $\frac{\delta\mathscr{S}}{\delta\mathbf{A}} = 0 = \mathbf{j} + \epsilon_0\ \dot{\mathbf{E}} - \frac{1}{\mu_0}\nabla\times\mathbf{B}$




## Axion electrodynamics

- Modified Lagrangian: $\mathscr{L}_\theta = 2\alpha\sqrt{\frac{\epsilon_0}{\mu_0}}\,\frac{\theta}{2\pi}\ \mathbf{E}\,.\,\mathbf{B}$

  with $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$ the fine structure constant

## Time Reversal Symmetry

- Time Reversal symmetry: $\theta = -\theta$ (mod $2\pi$)
  - Topological Insulator: $\theta = \pi$
  - Standard Insulator: $\theta = 0$
- In magnetic materials (no Time Reversal symmetry): $\theta$ arbitrary, and depends on $\mathbf{r}, t$

## Magnetic Weyl semimetal:

- For a single Weyl pair: $\theta(\mathbf{r}) = 2\mathbf{b}\,.\,\mathbf{r}$

# Magneto-electric effects in Weyl semi-metals

## Electrodynamics of an insulator

- Standard Maxwell Lagrangian (isotropic):

$$\mathscr{L}_0 = \frac{\epsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2 - \rho\phi + \mathbf{j}\,.\,\mathbf{A}$$

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  with $\alpha = \dfrac{e^2}{4\pi\epsilon_0\hbar c}$ the fine structure constant

- $$\nabla\,.\,\mathbf{E} = \frac{\rho}{\epsilon_0}\ -2c\alpha\nabla\left(\frac{\theta}{2\pi}\right).\,\mathbf{B}$$
- $$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\partial_t\mathbf{E}\ + \frac{2\alpha}{c}\nabla\left(\frac{\theta}{2\pi}\right)\times\mathbf{E}$$
- If $\theta$ is inhomogeneous: Maxwell eq. Modified !

# Magneto-electric effects in Weyl semi-metals

## Electrodynamics of an insulator

- Standard Maxwell Lagrangian (isotropic):
  $$\mathscr{L}_0 = \frac{\epsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2 - \rho\phi + \mathbf{j}\,.\,\mathbf{A}$$
- Action $\mathscr{S} = \int d^2\mathbf{r}dt\ \mathscr{L}$
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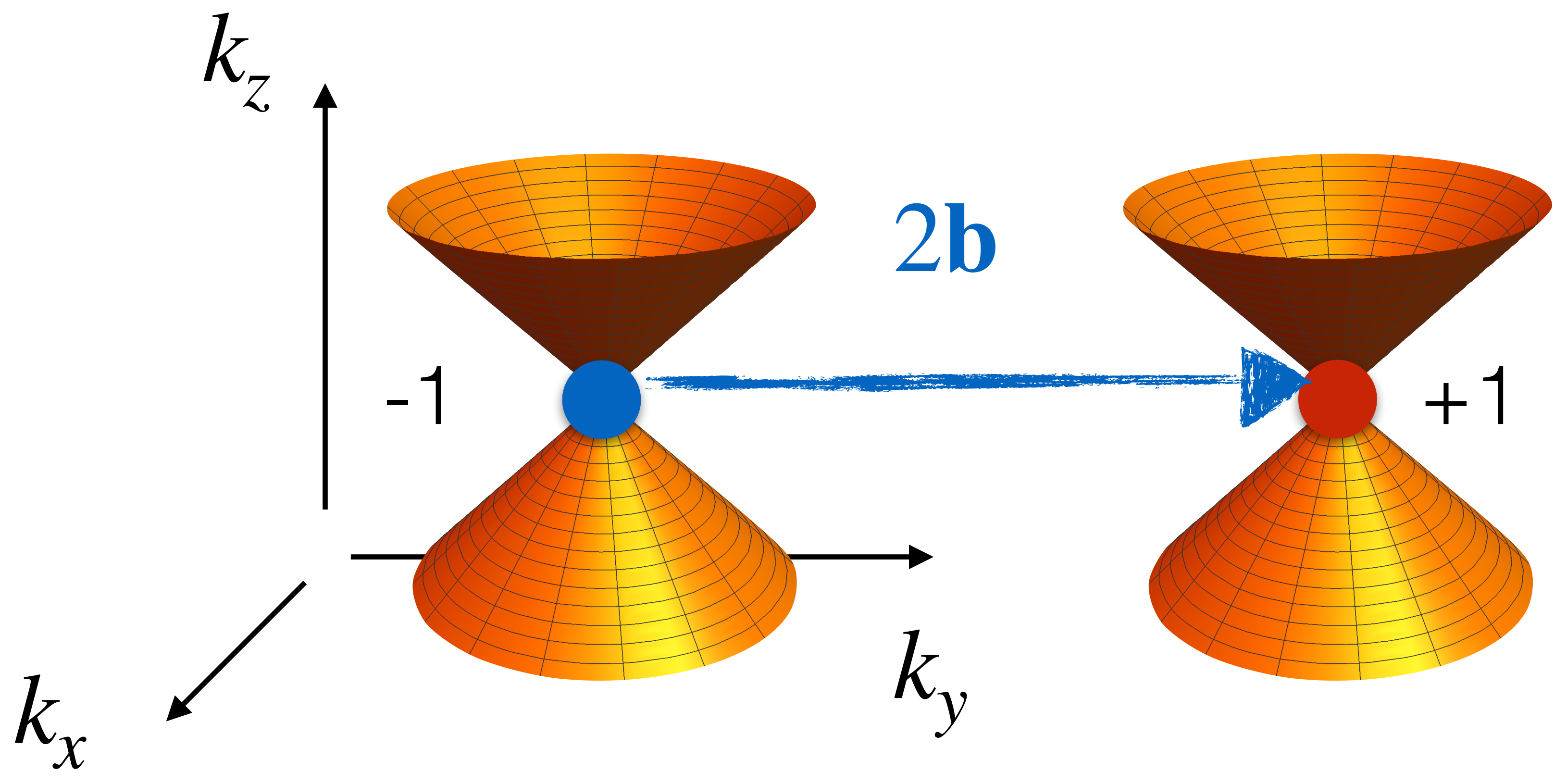


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  with $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$ the fine structure constant
- $\nabla\,.\,\mathbf{E} = \frac{\rho}{\epsilon_0}\ -2c\alpha\left(\frac{\mathbf{b}}{\pi}\right).\,\mathbf{B}$
- $\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\partial_t\mathbf{E}\ + \frac{2\alpha}{c}\left(\frac{\mathbf{b}}{\pi}\right)\times\mathbf{E}$

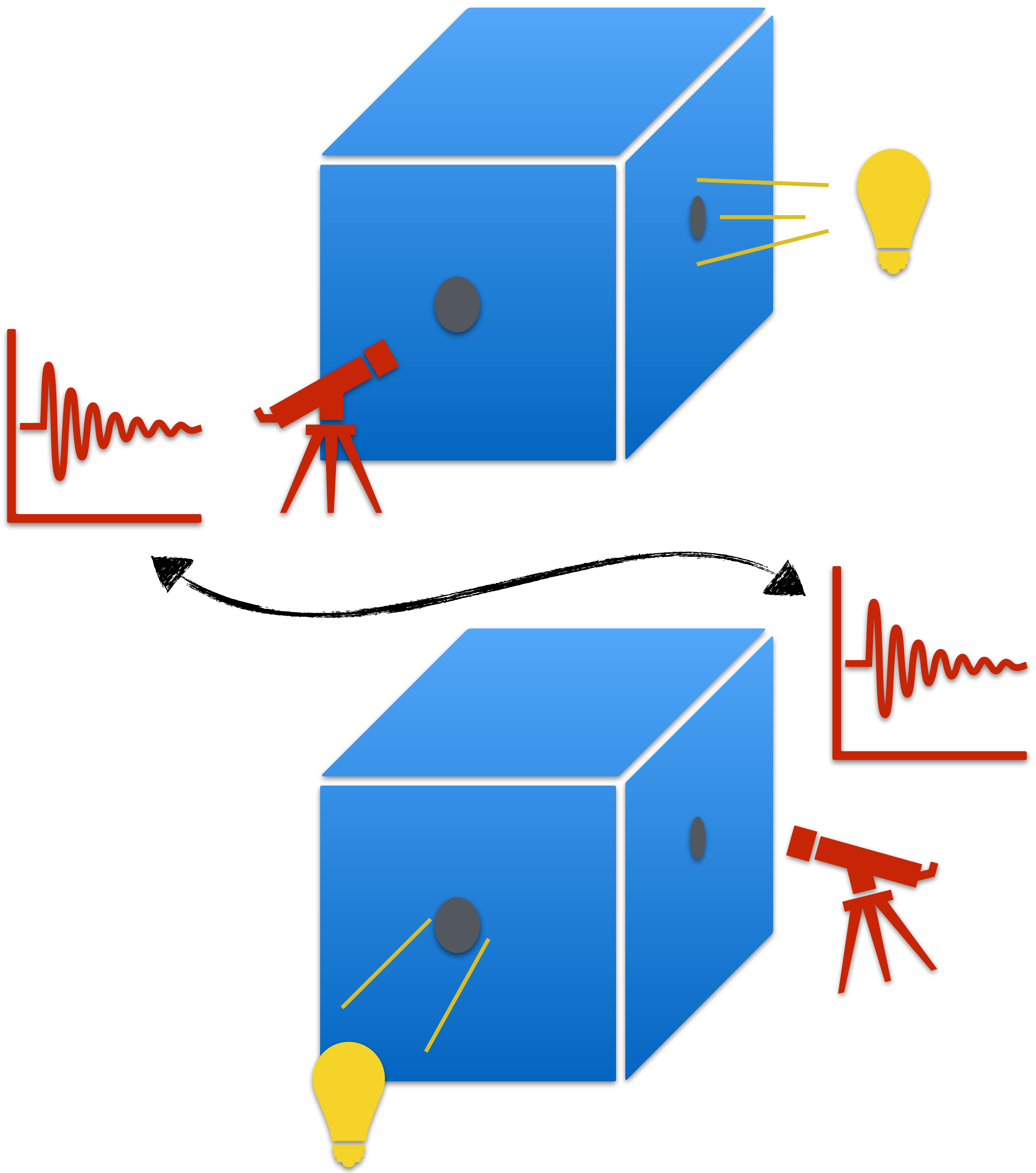
## Magnetic Weyl semimetal:

- For a single Weyl pair: $\theta(\mathbf{r}) = 2\mathbf{b}\,.\,\mathbf{r}$

# Nonreciprocity

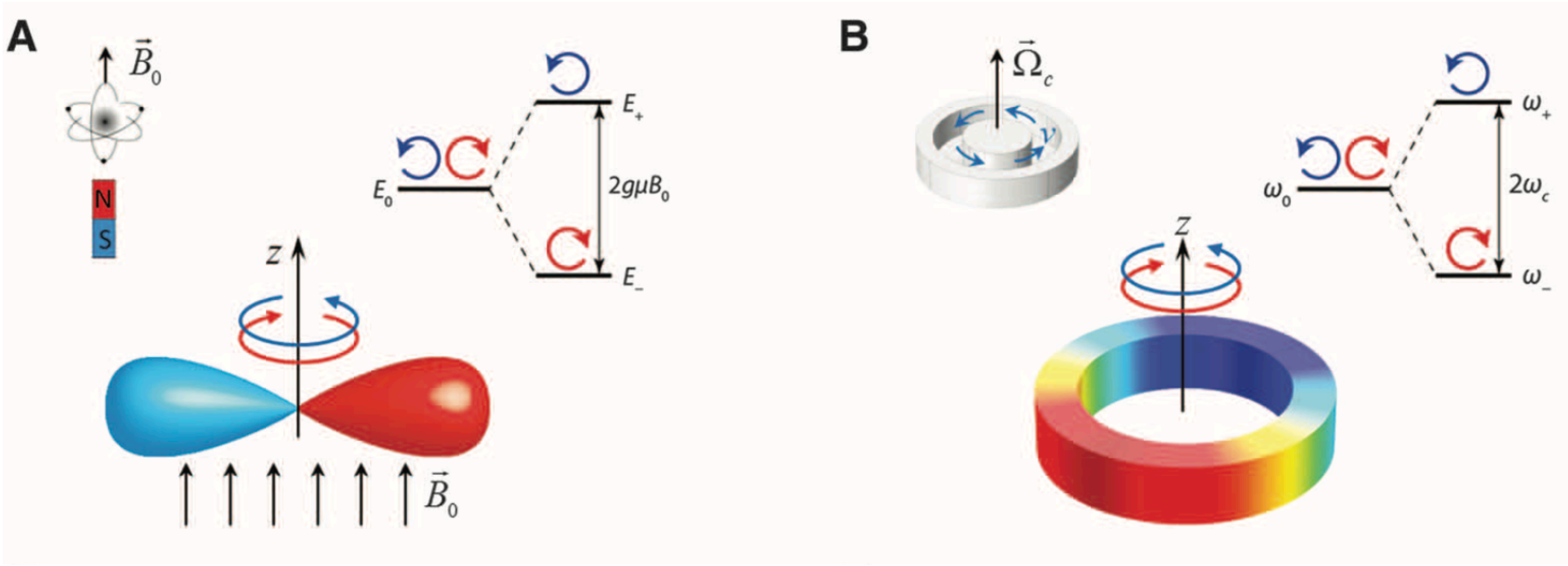
C. Caloz *et al*, Phys. Rev. Applied (2018)

A **nonreciprocal system** is defined as a system that exhibits **different transmitted fields** when its **source and detector are exchanged**



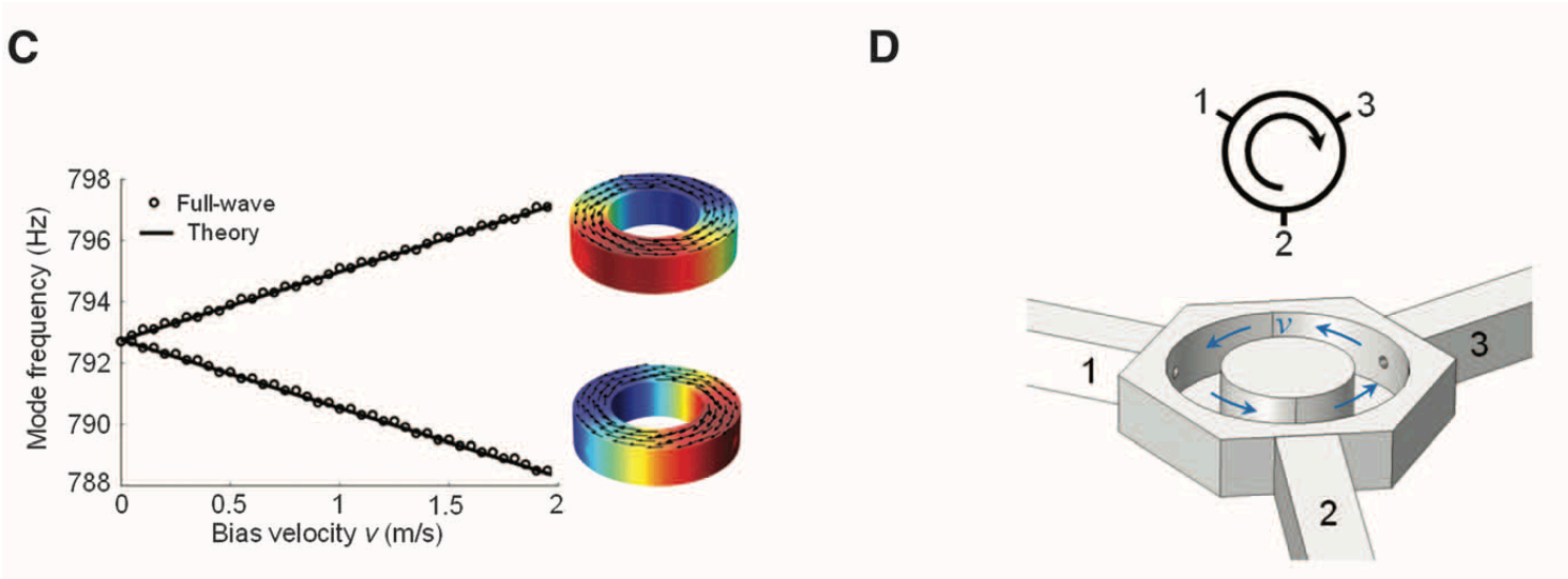
# Sound nonreciprocity

[R. Fleury *et al*, Science (2014)]

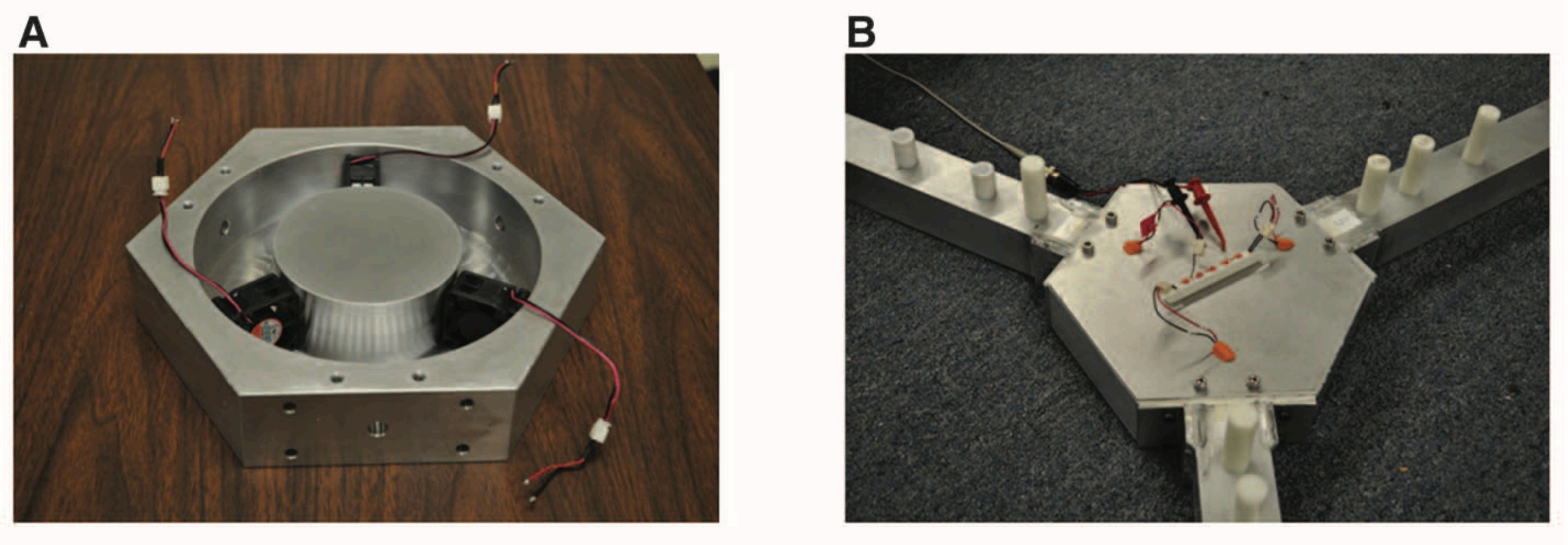



Standard Zeeman effect

Acoustic Zeeman effect

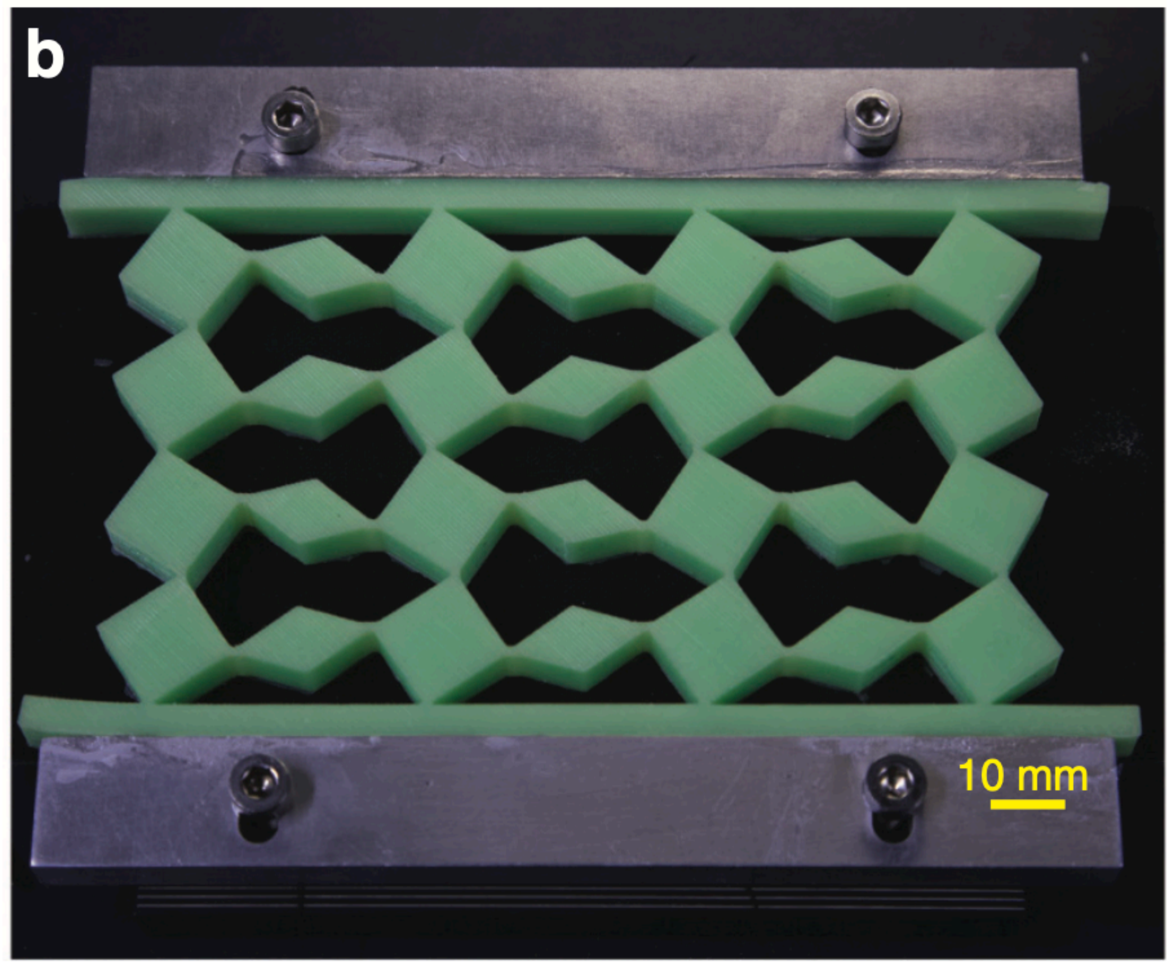


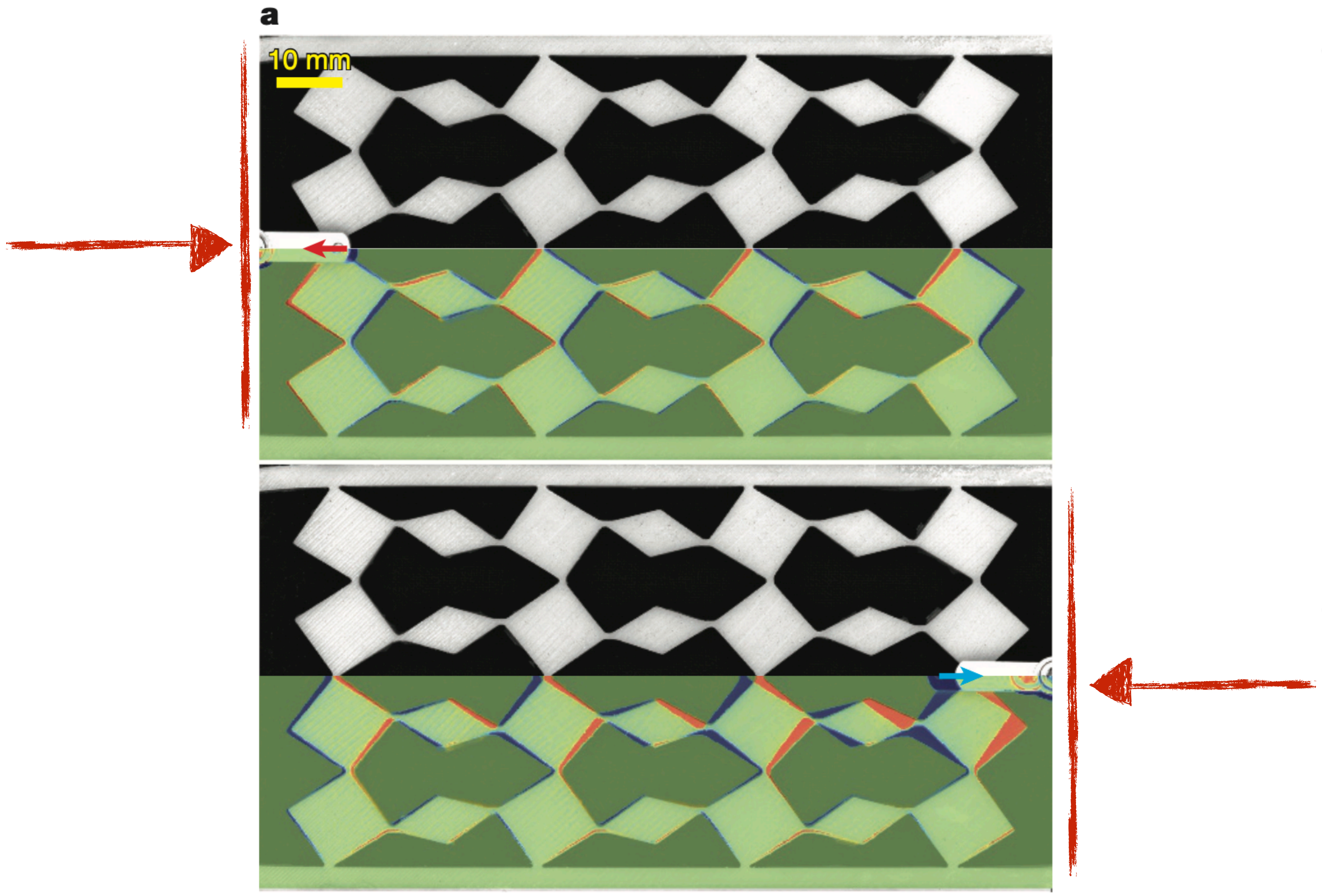

Acoustic circulator: 3-port implementation of acoustic Zeeman device

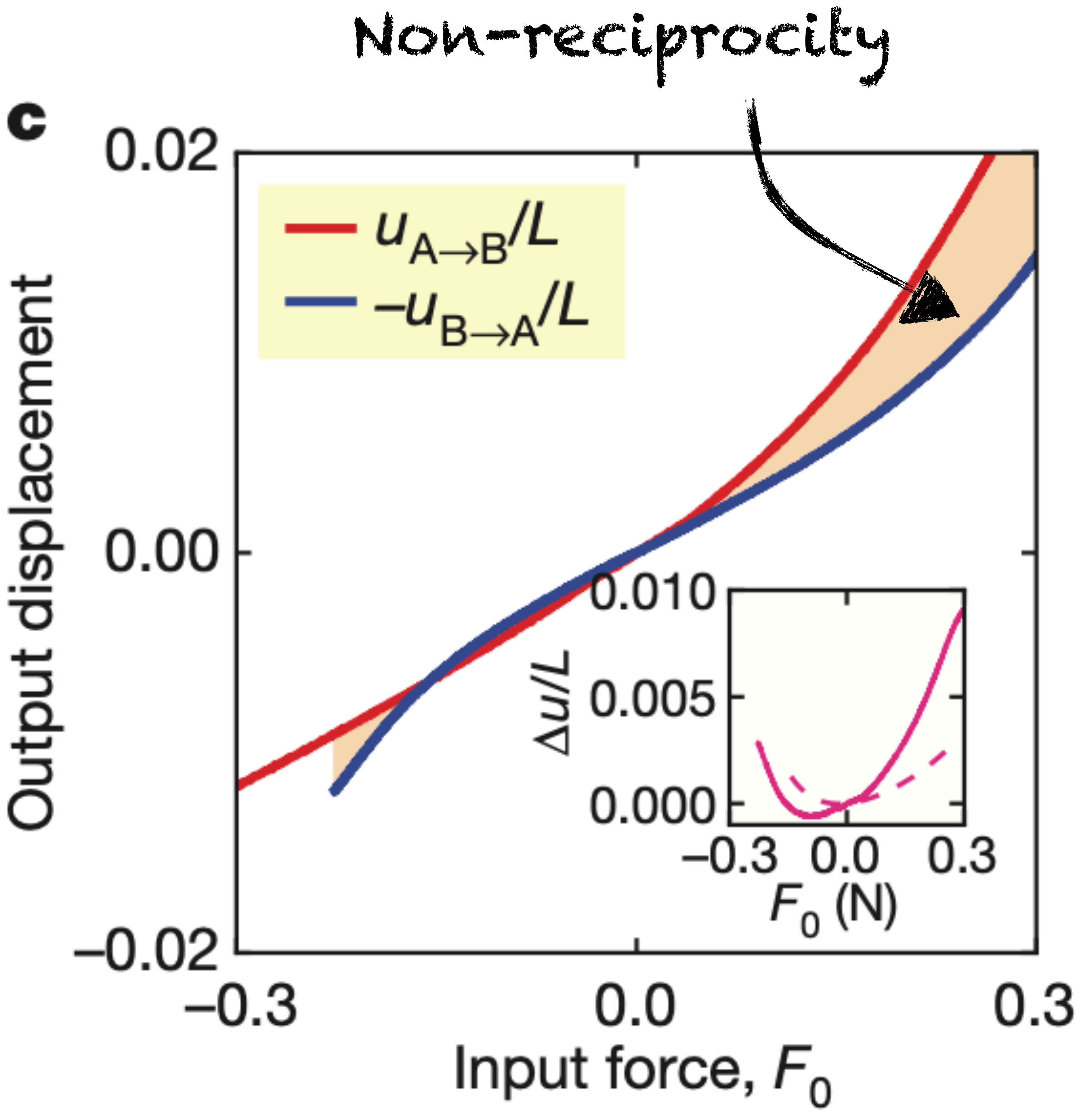


# Mechanical nonreciprocity

[C. Coulais *et al*, Nature (2017)]



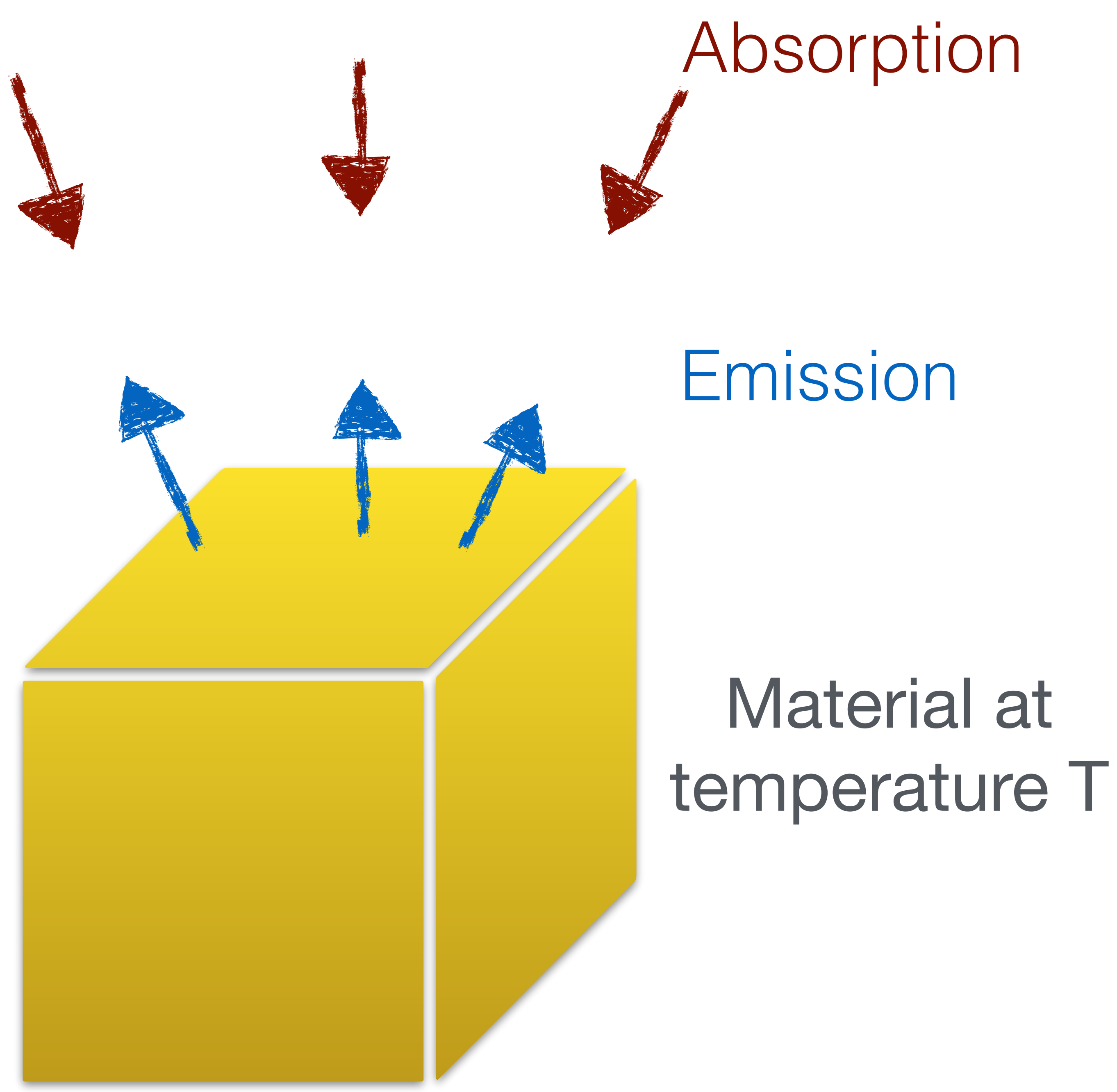




**2D topological mechanical metamaterial.**
For clarity, the image difference between the deformed and initial geometries have been overlaid on the bottom half of the pictures

# Electromagnetic nonreciprocity

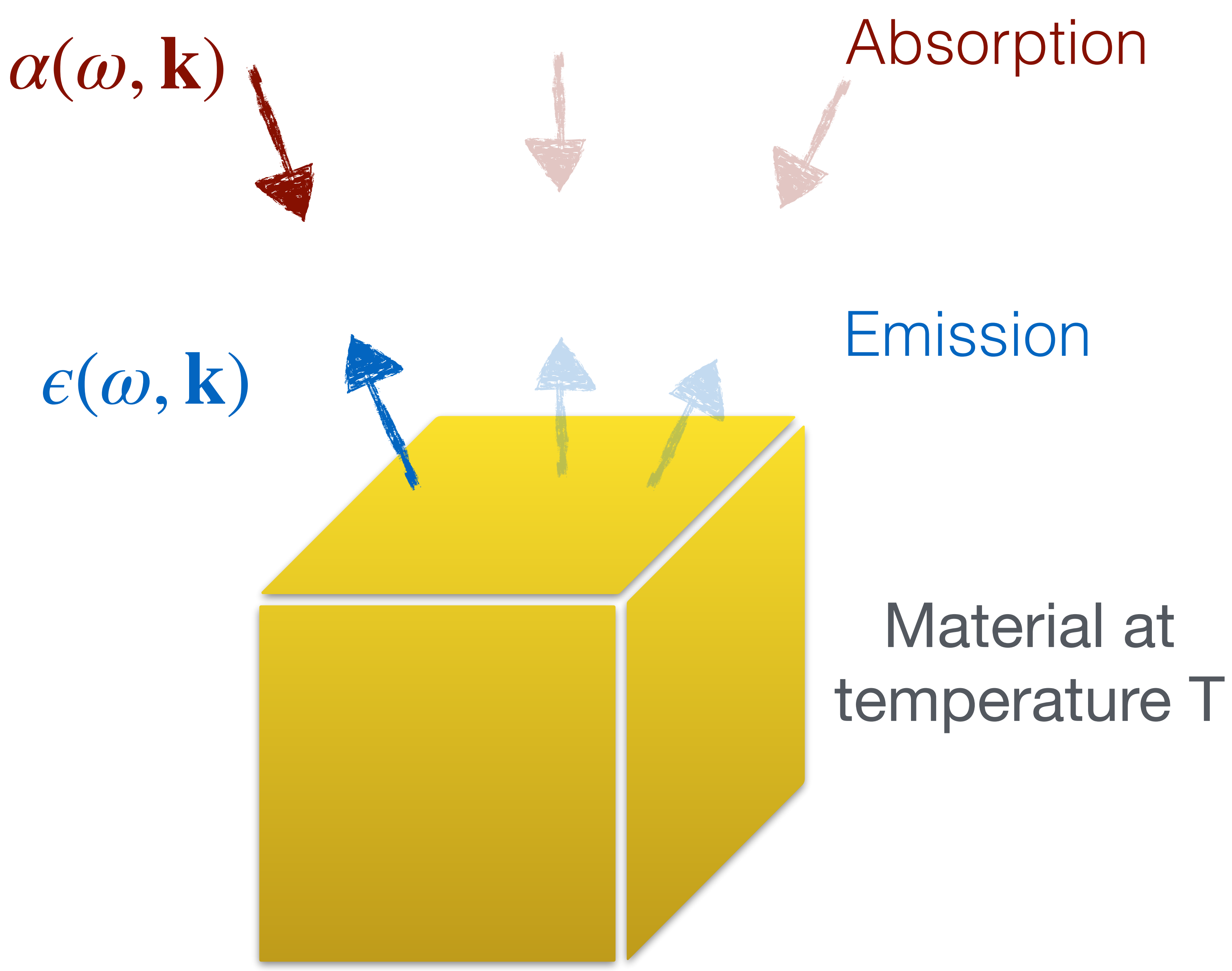
- **Equilibrium**
  - Vanishing net exchange of energy with environment
  - Black body radiation set by T




Tianji Liu *et al*, eLight (2022)

# Electromagnetic nonreciprocity

- **Equilibrium**
  - Vanishing net exchange of energy with environment
  - Black body radiation set by T
  - **Kirchoff law** (1860): absorptivity $\alpha(\omega, \mathbf{k})$ and emissivity $\epsilon(\omega, \mathbf{k})$ are equal, $\alpha(\omega, \mathbf{k}) = \epsilon(\omega, \mathbf{k})$
  - ~ valid away from equilibrium




Tianji Liu *et al*, eLight (2022)

# Electromagnetic nonreciprocity

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  - **Kirchoff law** (1860): absorptivity $\alpha(\omega, \mathbf{k})$ and emissivity $\epsilon(\omega, \mathbf{k})$ are equal, $\alpha(\omega, \mathbf{k}) = \epsilon(\omega, \mathbf{k})$
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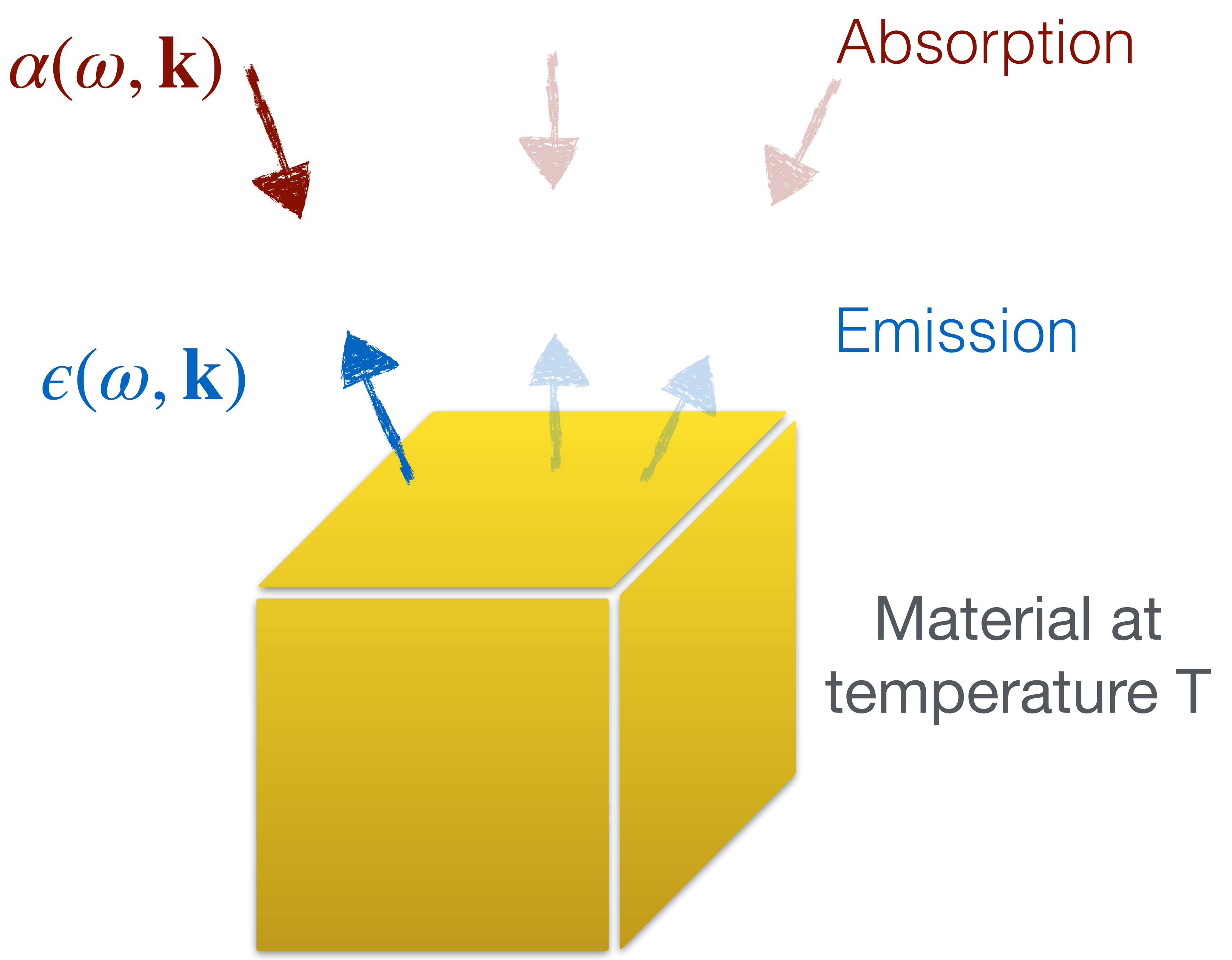
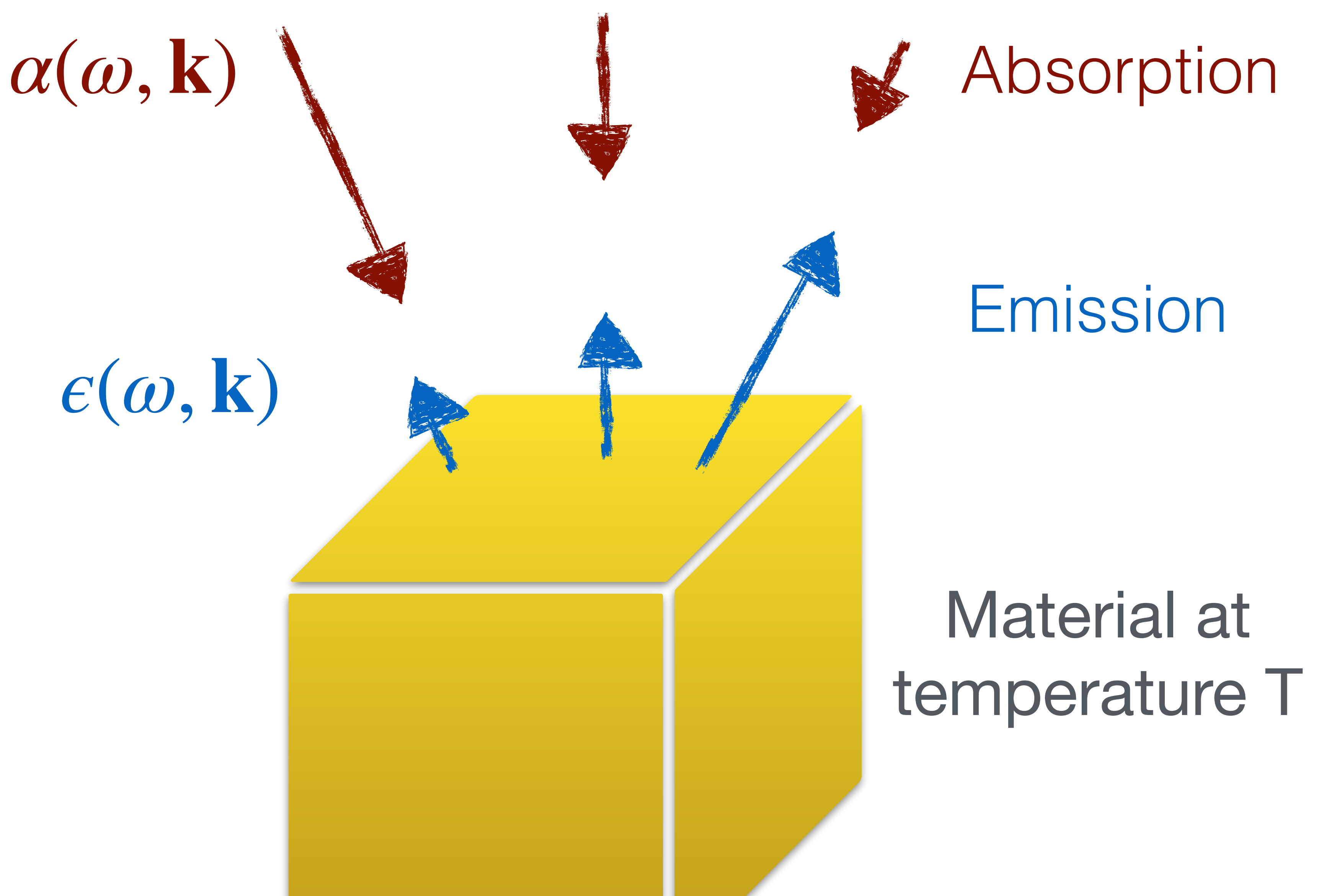



- **Optical non-reciprocal materials**
  - Assymetry in $\alpha(\omega, \mathbf{k}) \neq \epsilon(\omega, \mathbf{k})$
  - Increase of solar cell's efficiency
  - Passive radiative cooling under direct sunlight



# Electromagnetic nonreciprocity

Aaswath P. Raman *et al*, Nature (2014)

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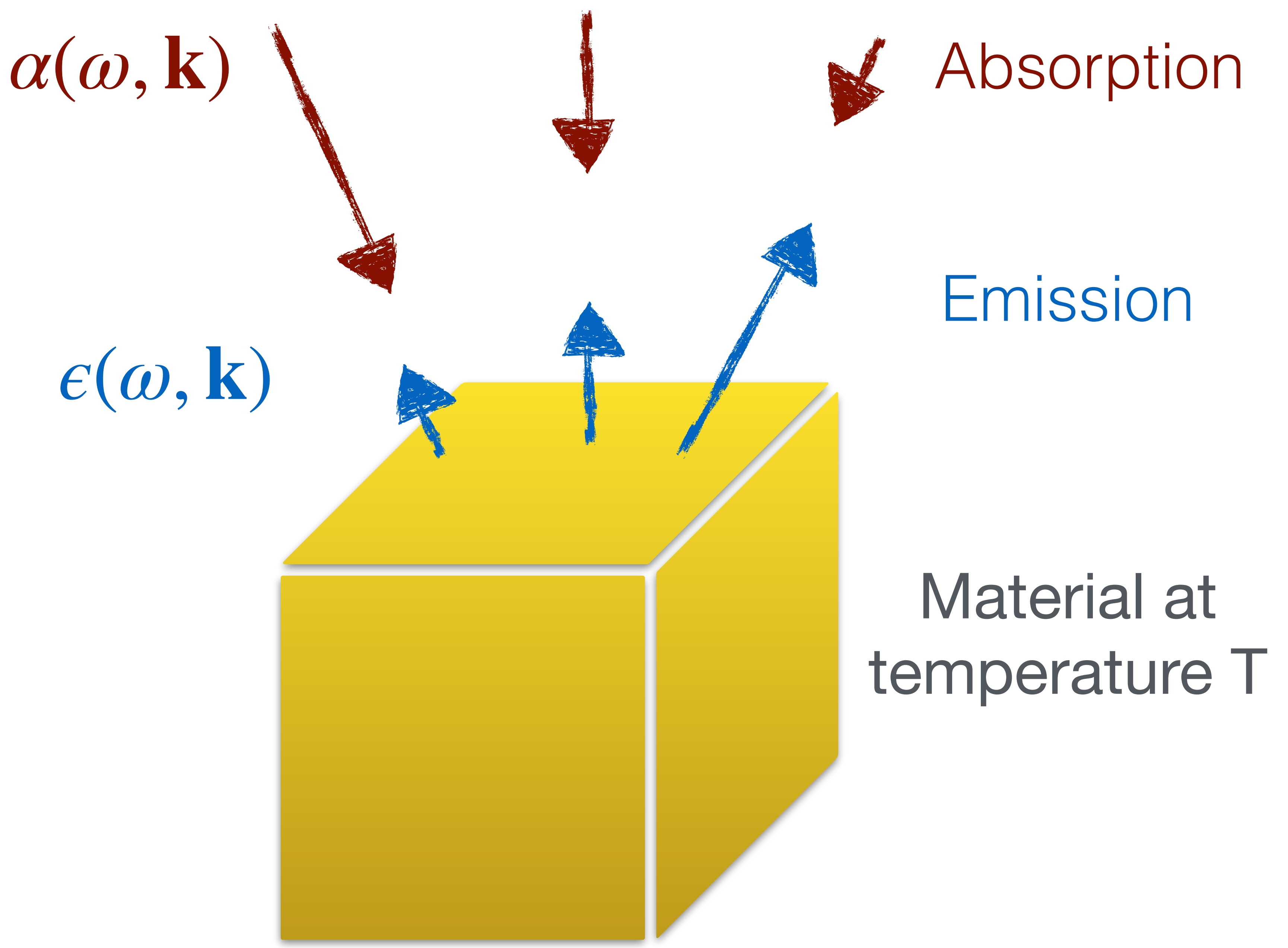
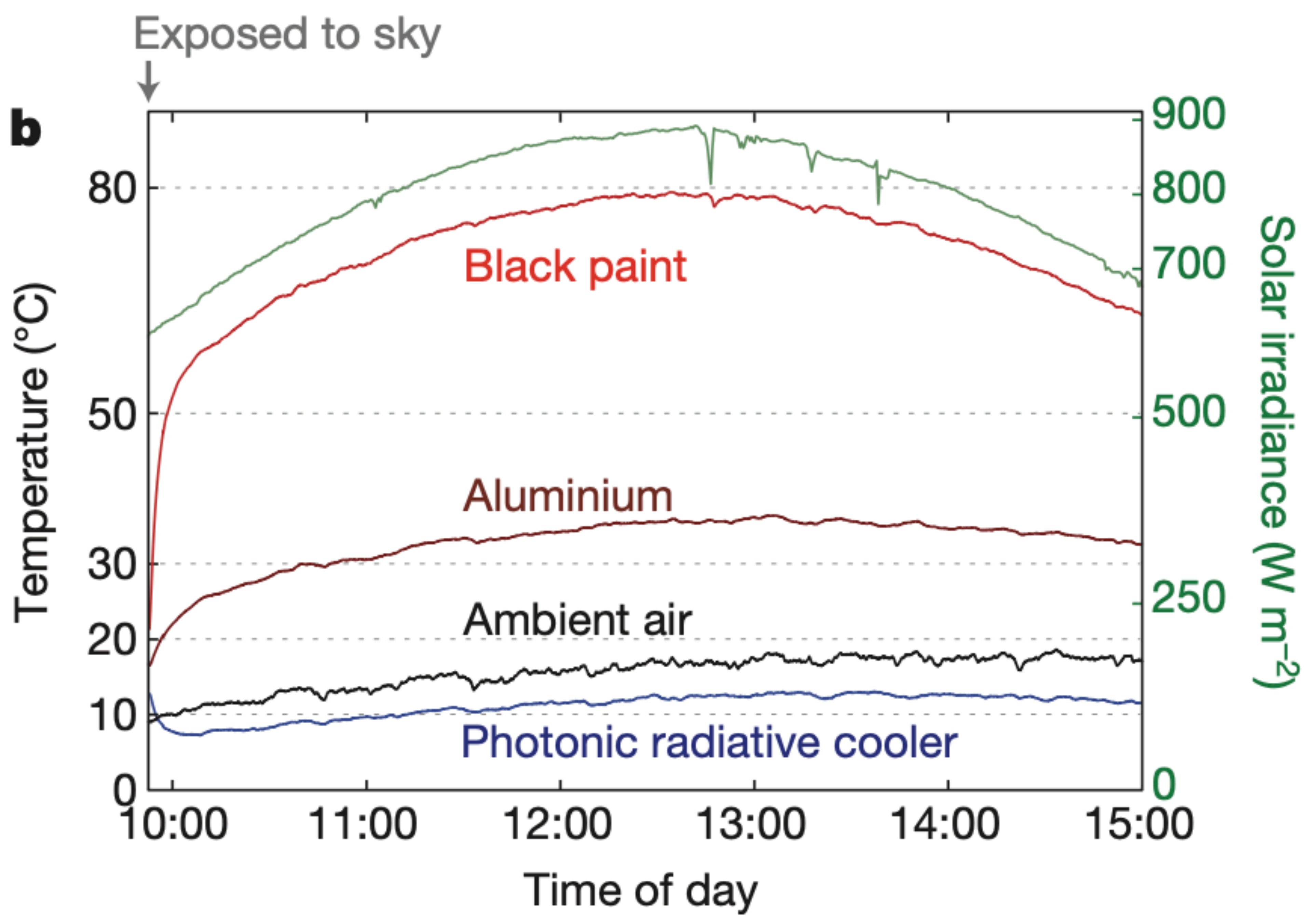
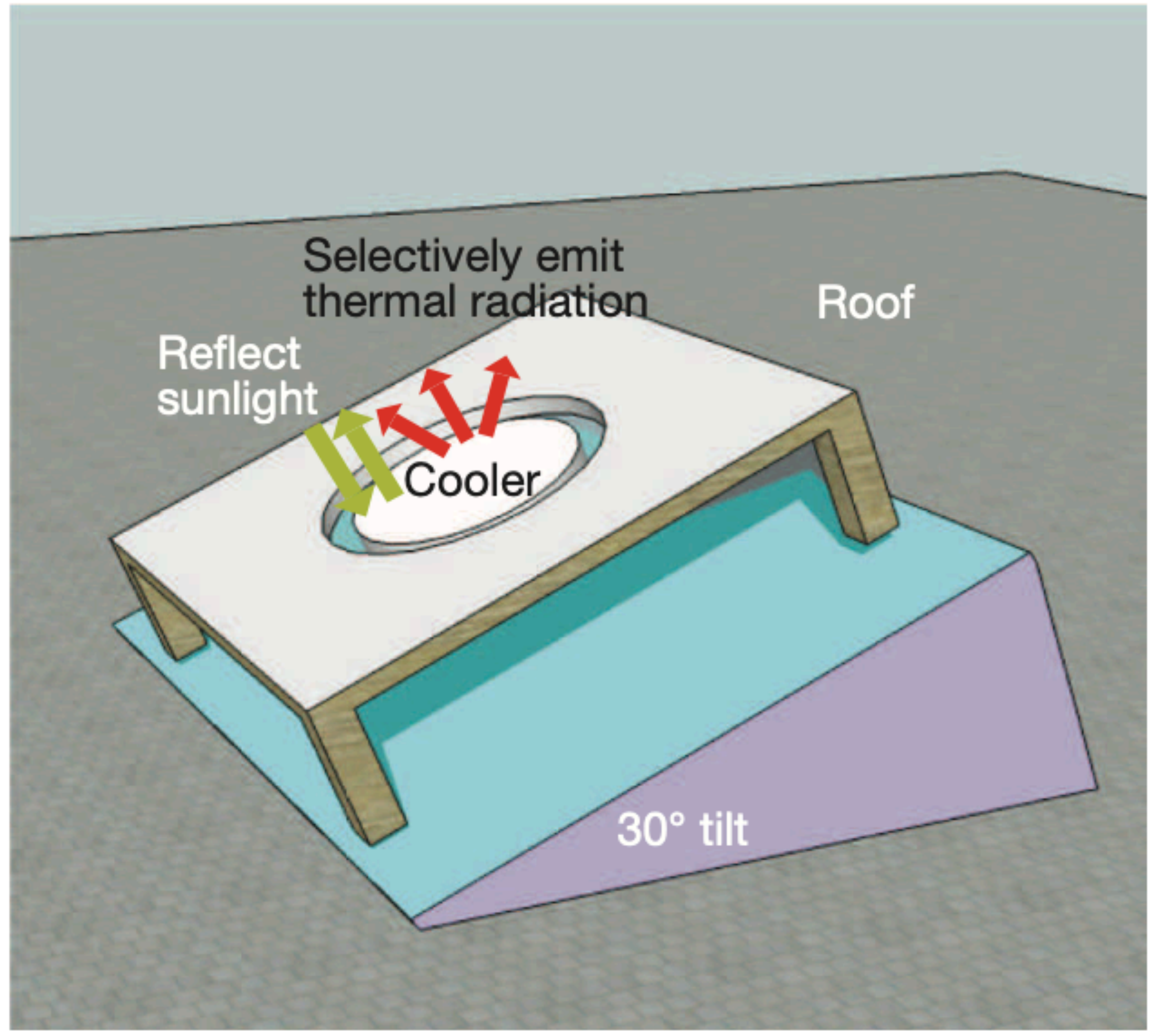


# Giant optical non-reciprocity in Weyl materials

Cheng Guo
*et al*, eLight (2023)

- **Optical non-reciprocal materials are rare**
  - Typically magneto-optical materials
  - Manifests itself as a asymmetric dielectric tensor: $\epsilon^T \neq \epsilon$
  - Measure non-reciprocity through $\gamma = \dfrac{|\epsilon - \epsilon^T|}{|\epsilon + \epsilon^T|}$
  - For magneto-optical materials, $\gamma \simeq \dfrac{\omega_c}{\omega}$ with the cyclotron frequency $\omega_c = \dfrac{eB}{m*}$
  - For $B \sim 1$ T, $\omega_c \sim 1$ THz, we get $\gamma \sim 10^{-3} - 10^{-2}$ at optical frequencies (weak non-reciprocity)

- **Magnetic Weyl semimetals**
  - Potential for $\gamma \sim 1$ at optical frequencies (giant non-reciprocity)

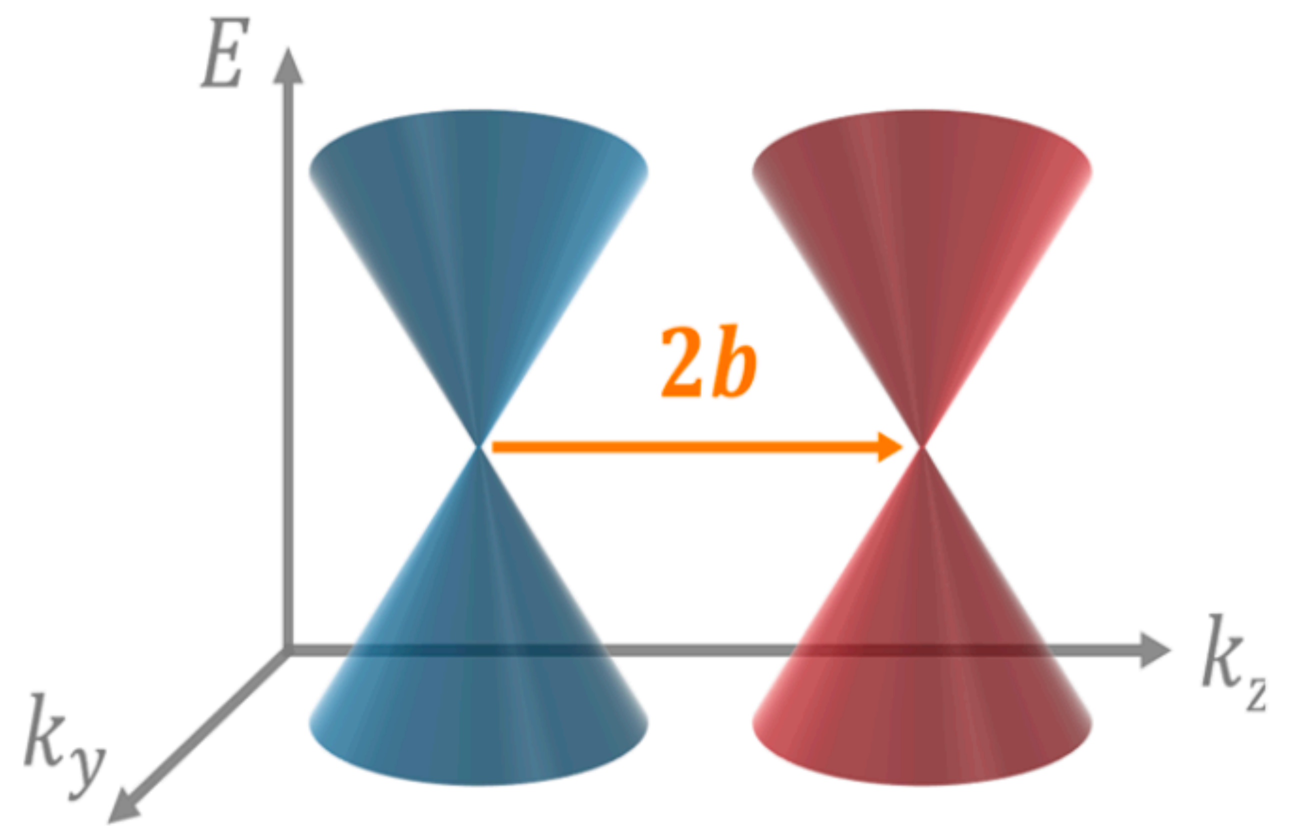
# Giant optical non-reciprocity

- **Dielectric tensor for a magnetic Weyl semimetal with $\mathbf{b} \parallel z$**
  - From $\epsilon_D(\omega) = \epsilon_b(\omega) + \frac{i}{\omega}\sigma(\omega)$

Background permittivity — conductivity tensor of Weyl electrons

We get $\epsilon = \begin{pmatrix} \epsilon_D & i\epsilon_a & 0 \\ -i\epsilon_a & \epsilon_D & 0 \\ 0 & 0 & \epsilon_D \end{pmatrix}$, typical form of a gyrotropic medium

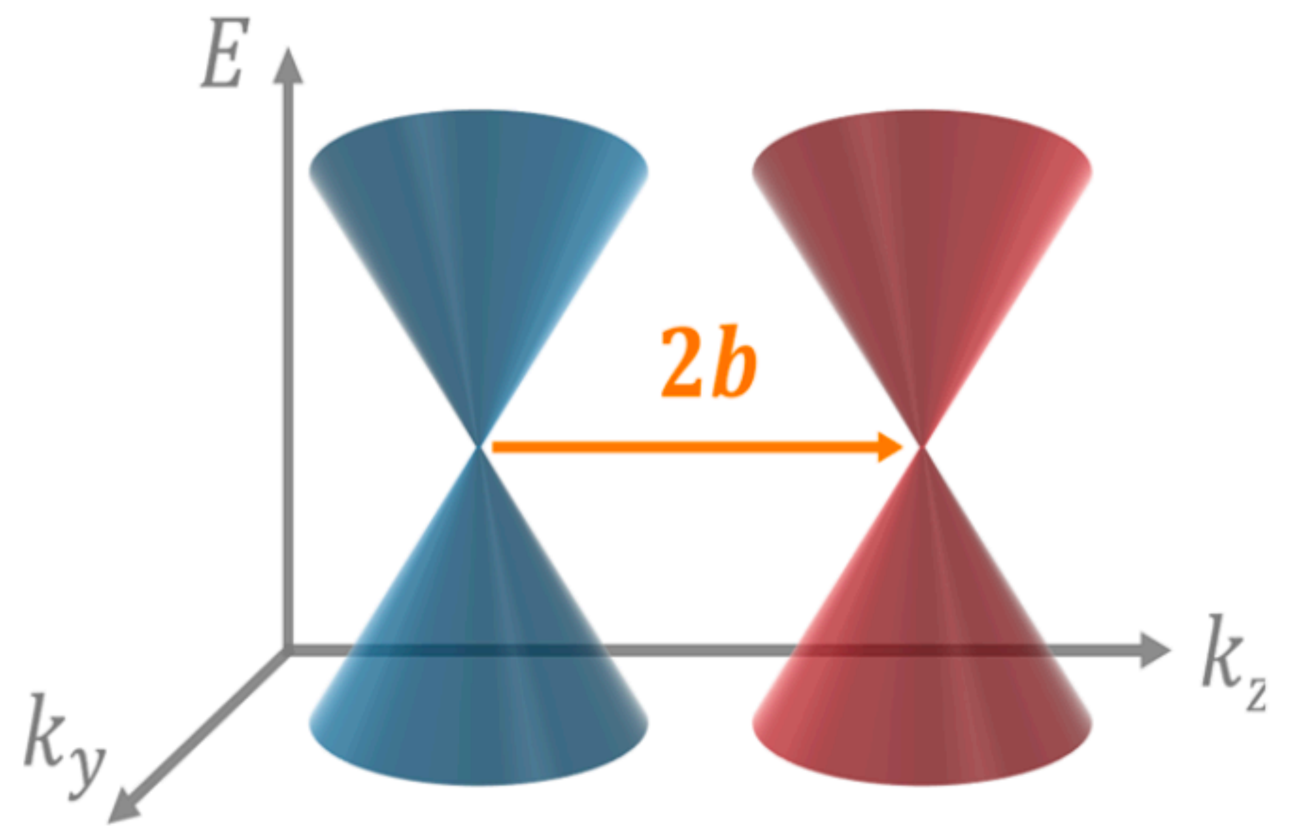
$\epsilon_a(\omega) = \frac{2b}{2\pi}\frac{1}{\omega}\frac{e^2}{h}$ (anomalous quantum Hall effect)



# Giant optical non-reciprocity

O. V. Kotov and Y. E. Lozovik, PRB (2018)

- **Dielectric tensor for a magnetic Weyl semimetal with $\mathbf{b} \parallel z$**




  - From $\epsilon_D(\omega) = \epsilon_b(\omega) + \frac{i}{\omega}\sigma(\omega)$

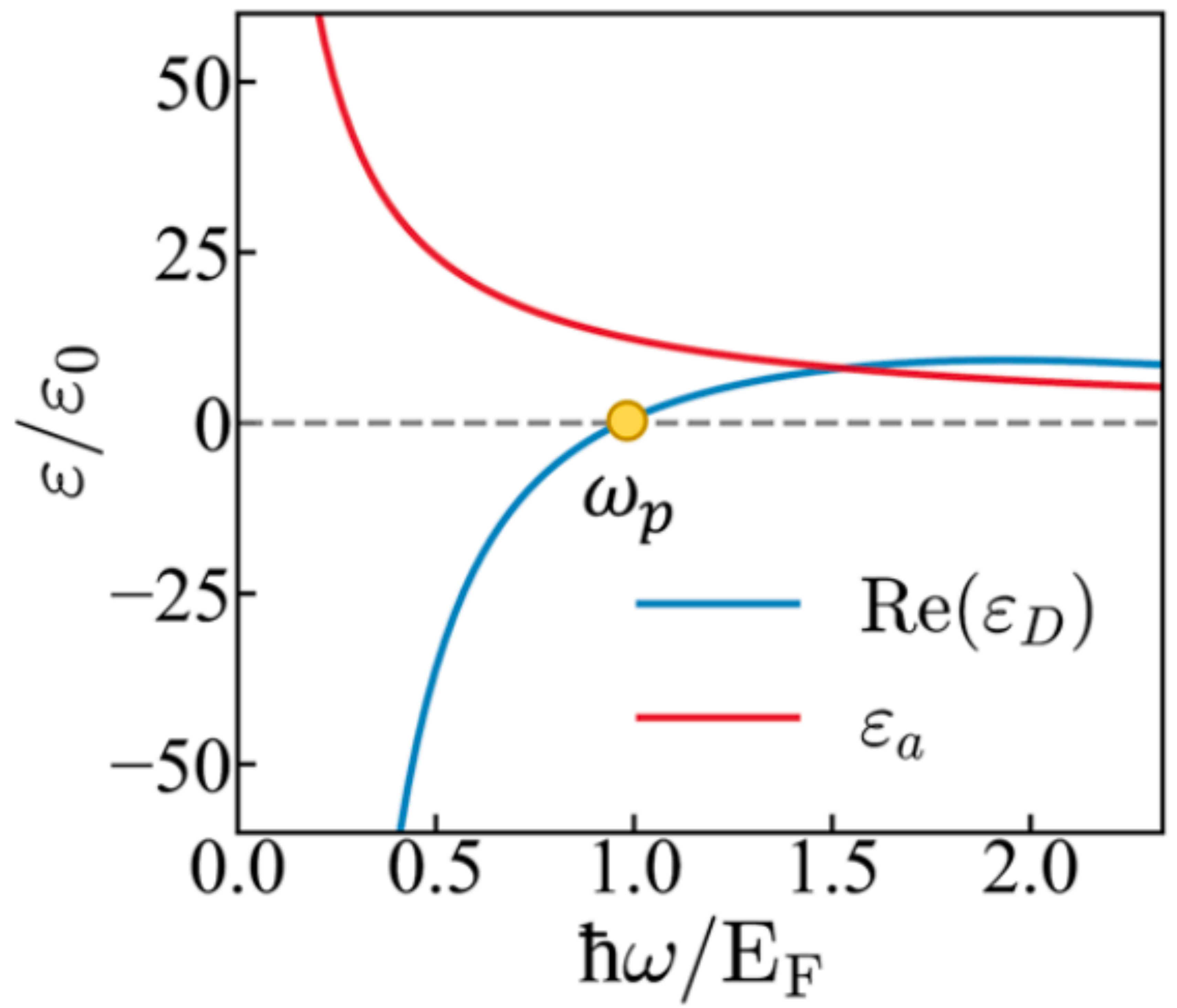
We get $\epsilon = \begin{pmatrix} \epsilon_D & i\epsilon_a & 0 \\ -i\epsilon_a & \epsilon_D & 0 \\ 0 & 0 & \epsilon_D \end{pmatrix}$, typical form of a gyrotropic medium

$$\epsilon_a(\omega) = \frac{2b}{2\pi}\frac{1}{\omega}\frac{e^2}{h}$$ (anomalous quantum Hall effect)

  - From Kubo formula:

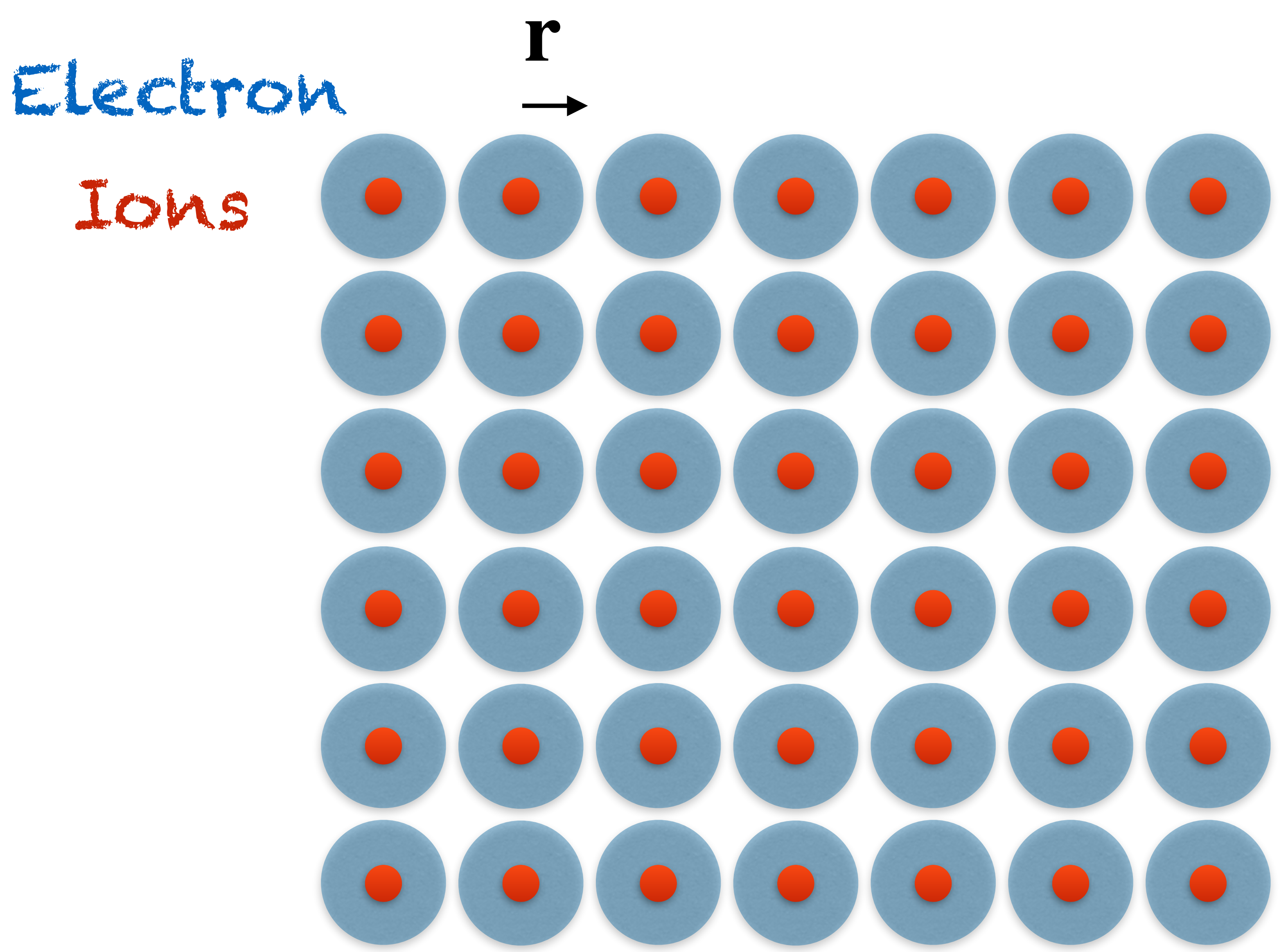
$$\sigma_D(\omega) = \frac{e^2}{h}\frac{k_F}{6}\left[\Omega\Theta(\Omega-2) + \frac{i}{\pi}\left(\frac{4}{\Omega} - \Omega \ln \frac{4\epsilon_C^2}{|\Omega^2 - 4|}\right)\right]$$ with

$\Omega = \hbar(\omega + i\tau^{-1})/E_F$ , $\epsilon_c = E_C/E_F$ (cut-off)



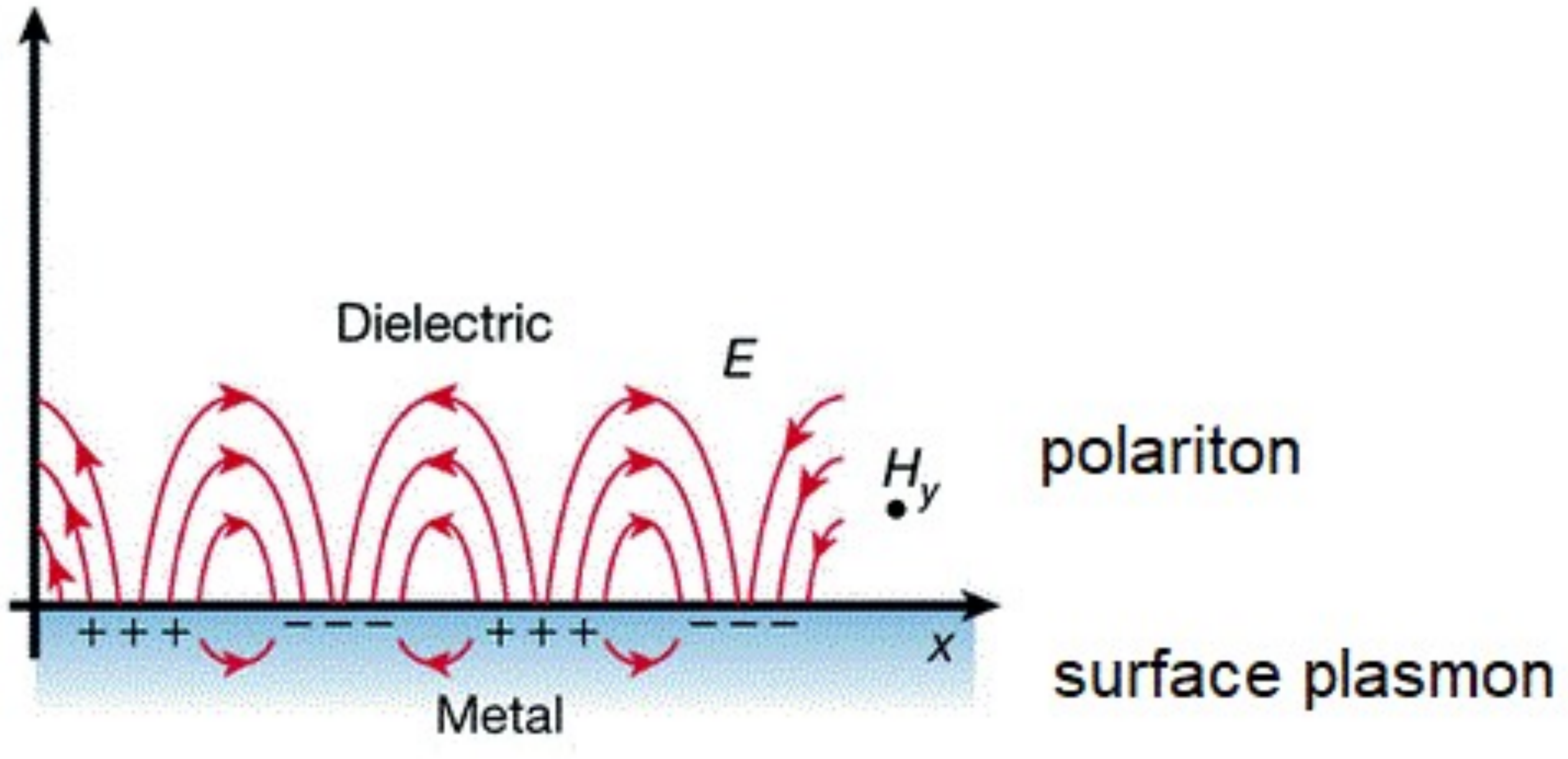

  - Non-reciprocity $\gamma \simeq |\epsilon_a/\epsilon_D| \sim 1$ over wide frequency range

# Non-reciprocal Waves at the surface




- **Plasmon**
  - Displacement $\mathbf{r}$ creates polarization $\mathbf{P} = nq\mathbf{r}$
  - Induces electric field $\mathbf{E} = -4\pi\mathbf{P}$
  - Restoring force: $m\partial_t^2\mathbf{r} = q\mathbf{E} = -4\pi ne^2\mathbf{r}$ (harmonic oscillator)
  - Plasma frequency: $\Omega_p^2 = \dfrac{4\pi ne^2}{m}$
  - Quantum: plasmon

- **Surface Plasmon polariton**
  - Collective electromagnetic and electron-charge excitations confined to the surface of a metal or semiconductor
  - Fields of the form $\mathbf{E} = \mathbf{E}_0 e^{iq_x x + q_y y} e^{-i\omega t} e^{-\kappa|z|}$



# Non-reciprocal Waves at the surface

J. Hofmann and S. Das Sarma PRB (2016)

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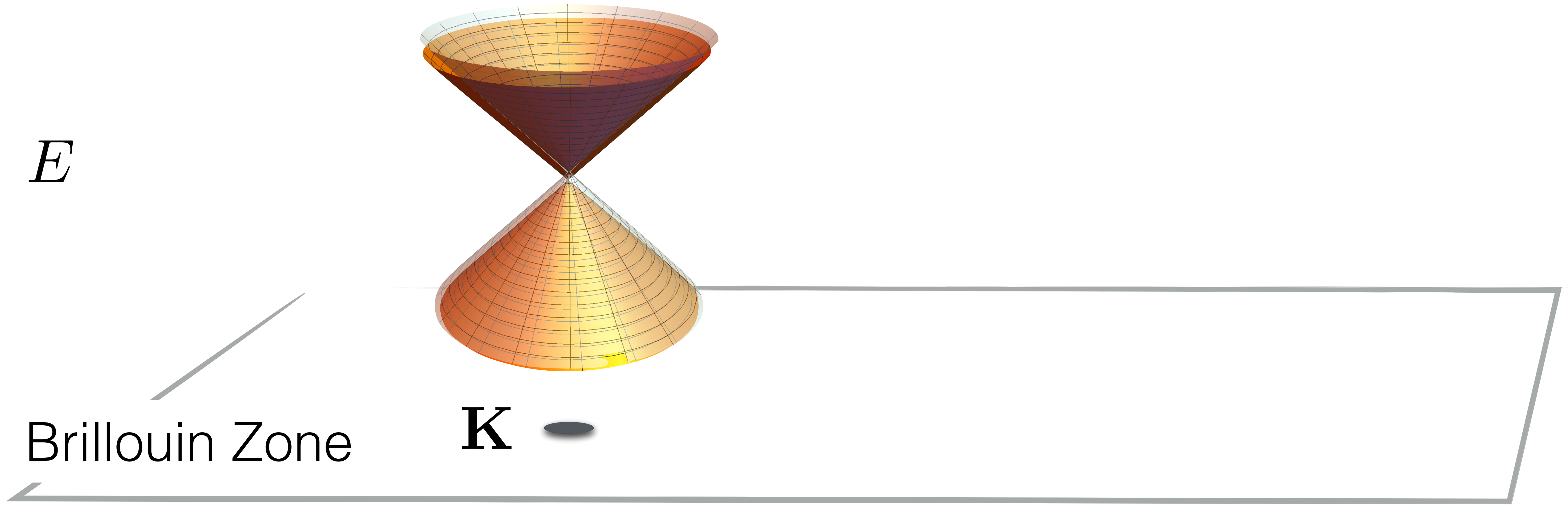
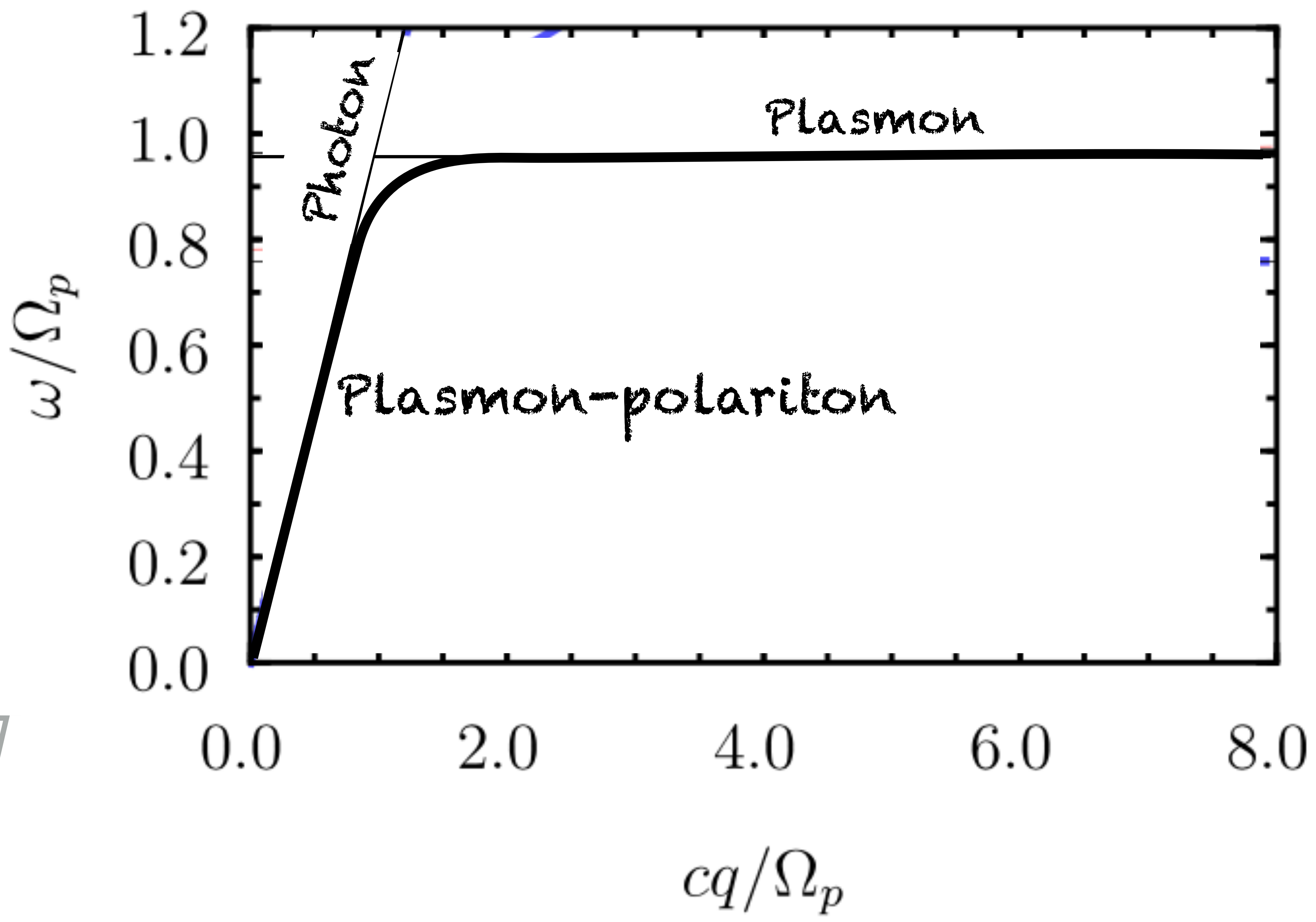
  - Fields of the form $\mathbf{E} = \mathbf{E}_0 e^{iq_x x + q_y y} e^{-i\omega t} e^{-\kappa|z|}$

Plasmon frequency: $\Omega_p^2 = \frac{4\alpha}{3\pi}\left(\frac{\mu}{\hbar}\right)^2$

with $\alpha = \frac{e^2}{\hbar v_F \epsilon_\infty}$

- **Surface Plasmon polariton in semi-metals**
  - Dirac semimetal (2 overlapping Weyl cones)



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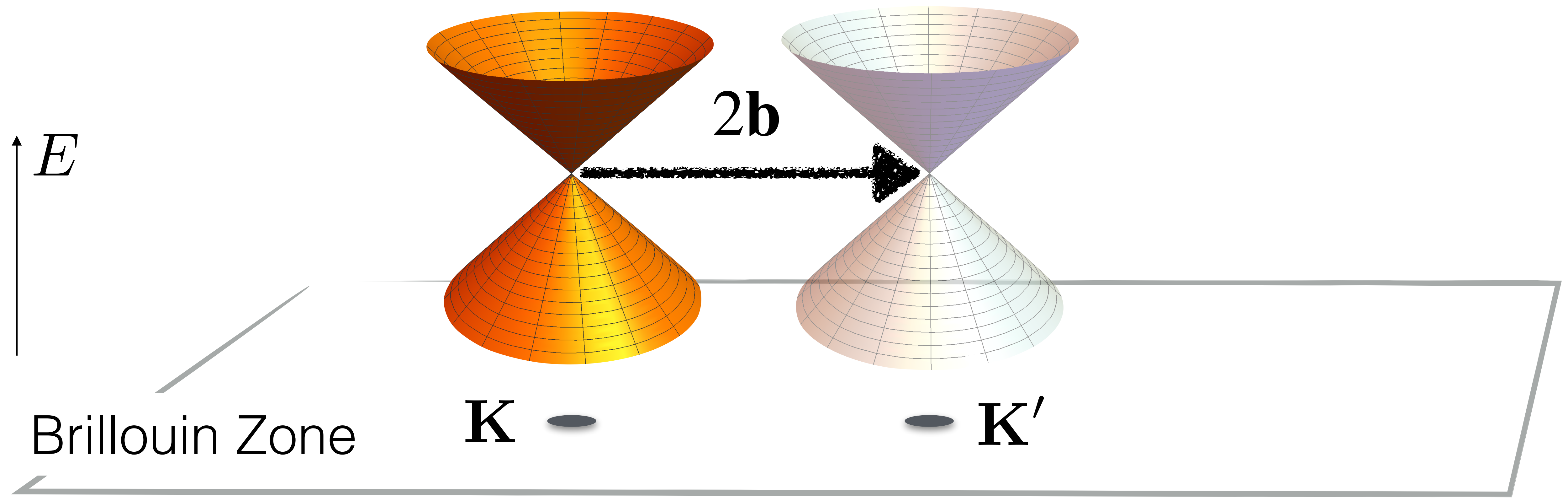
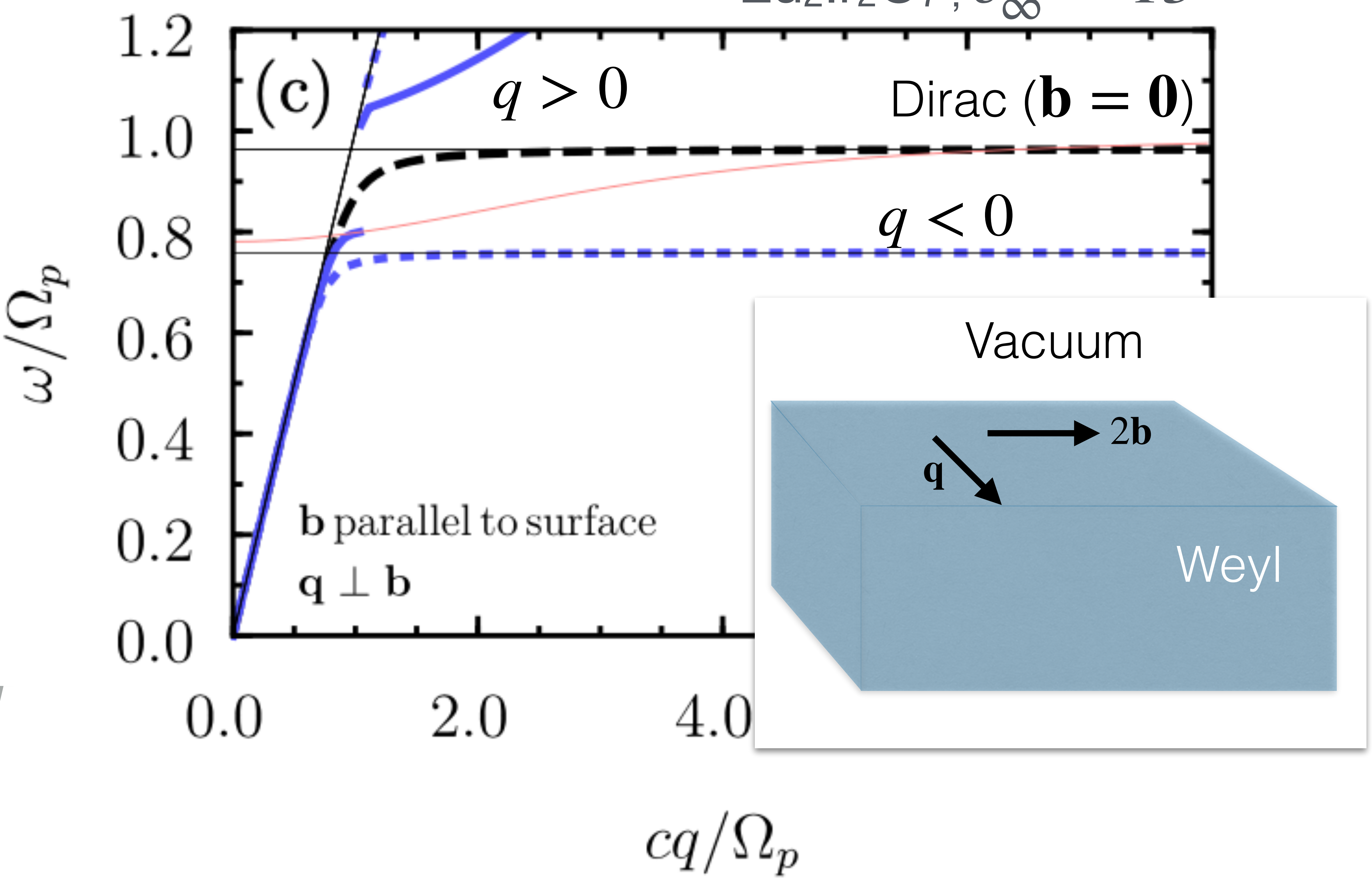
- **Surface Plasmon polariton in semi-metals**
  - Dirac semimetal (2 overlapping Weyl cones)
  - Weyl semimetal (distance $2\mathbf{b}$ between cones): non-reciprocal



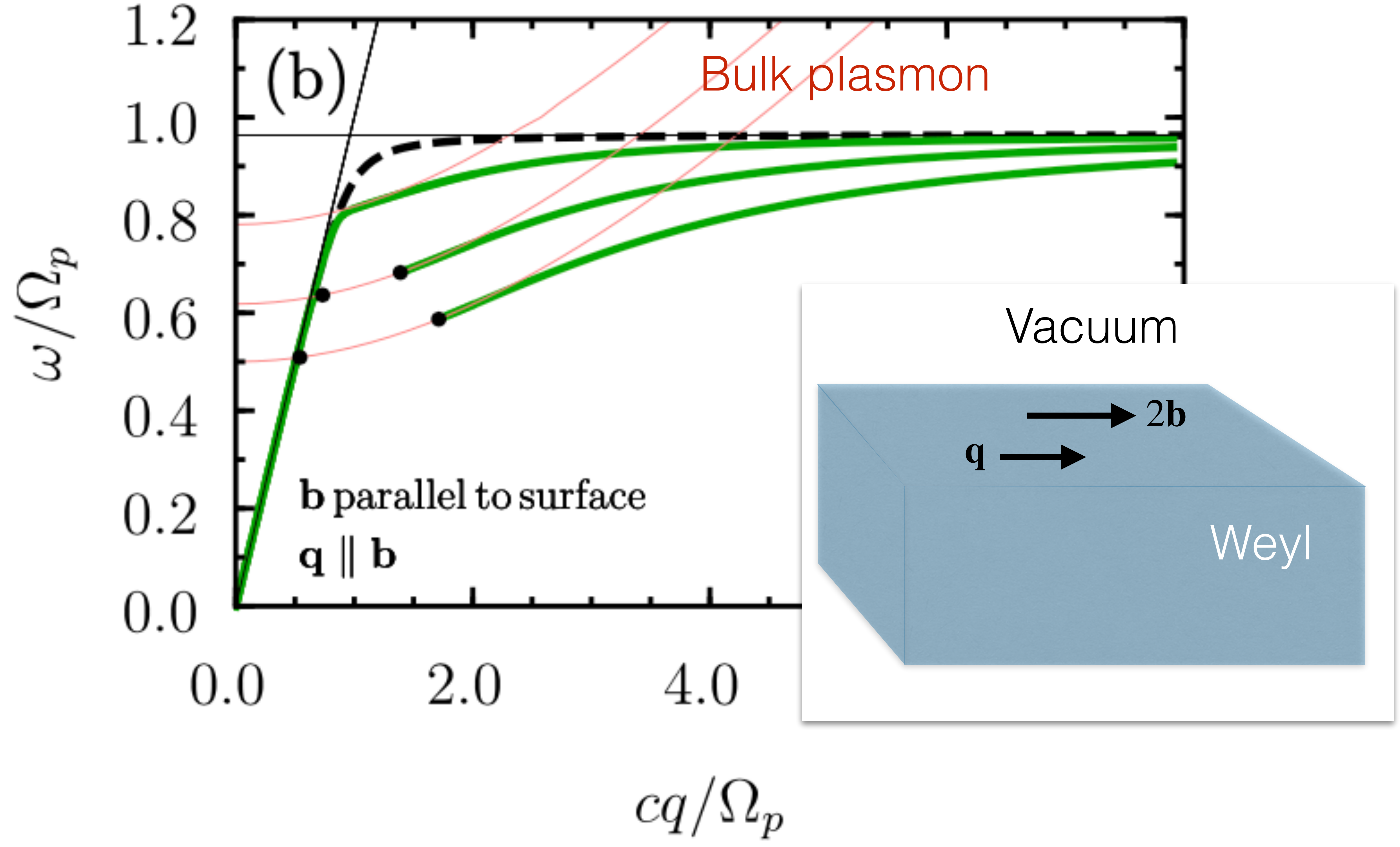

$Eu_2Ir_2O_7$, $\epsilon_\infty = 13$

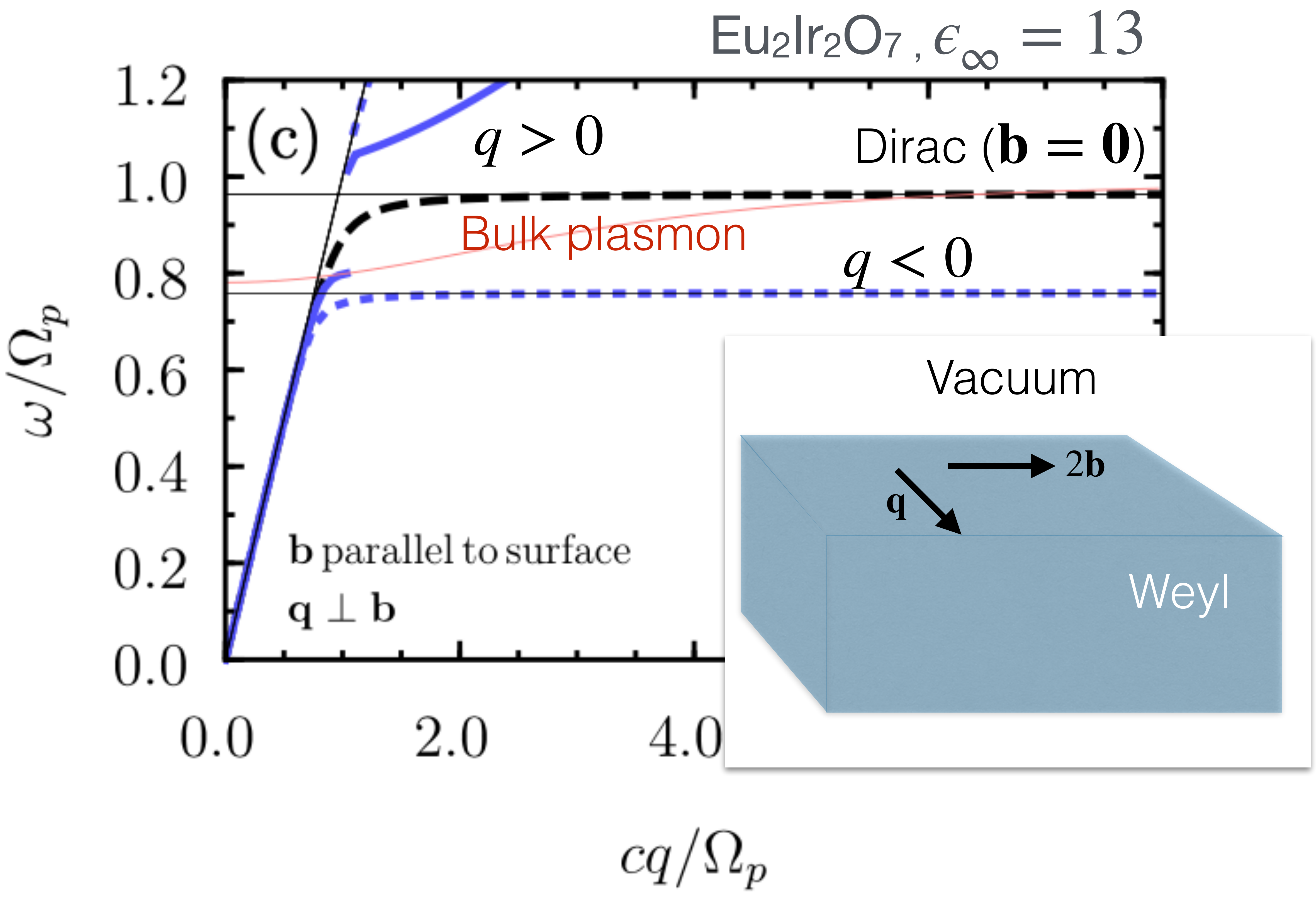


# Non-reciprocal Waves at the surface

J. Hofmann and S. Das Sarma
PRB (2016)

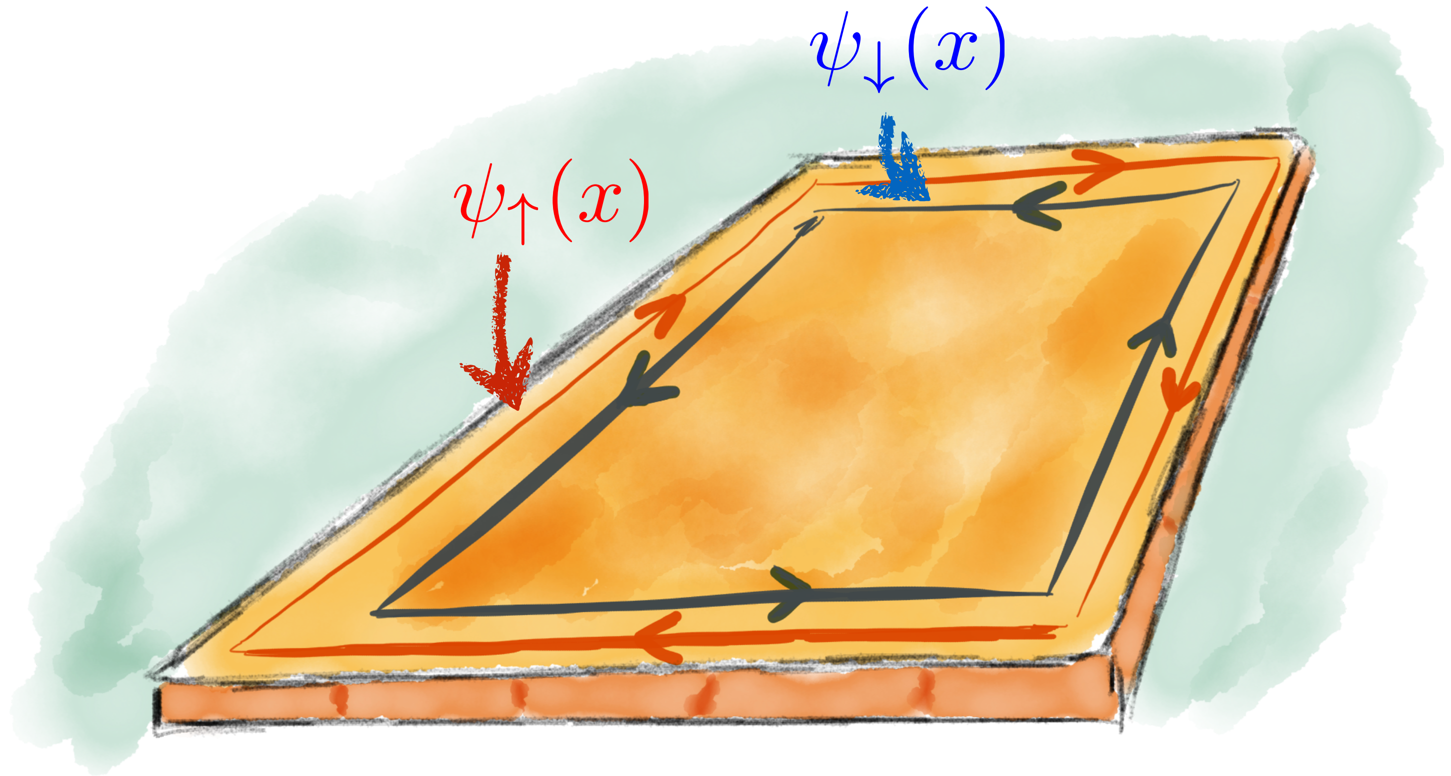
- **Surface Plasmon polariton in semi-metals**
  - Dirac semimetal (2 overlapping Weyl cones)
  - Weyl semimetal (distance $2\mathbf{b}$ between cones): non-reciprocal / reciprocal

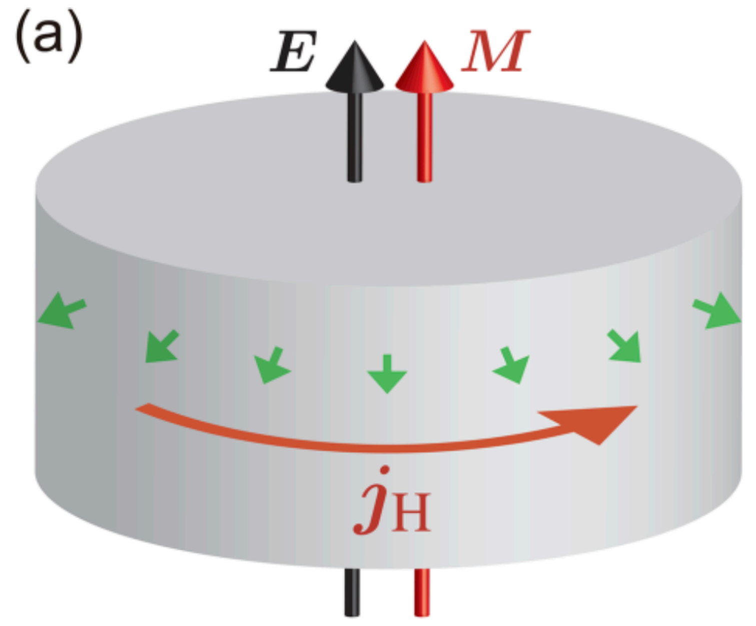


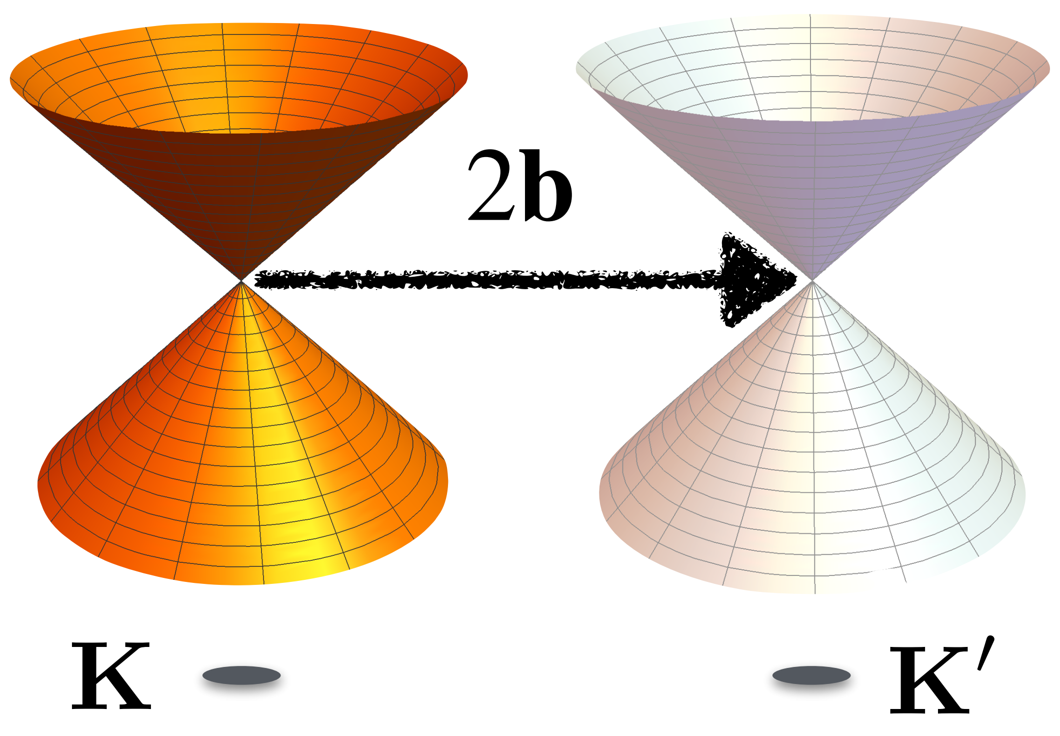



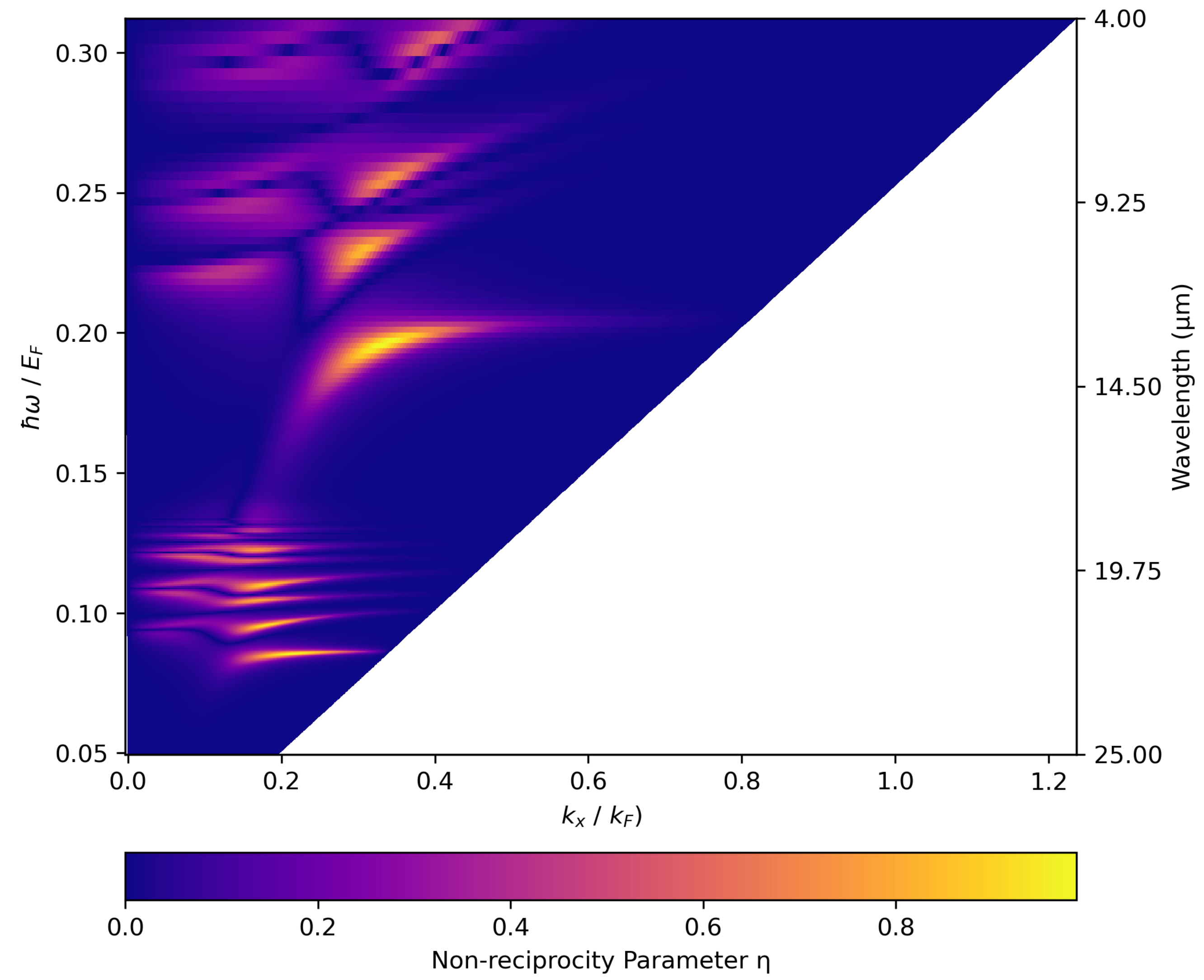
# Recap

- **2D topological insulators**
  - Edge states versus bulk: the Quantum Hall Effect example
  - Modified electrodynamics and quantized Hall response in 2D
- **3D topological insulators**
  - Surface properties
  - Modified electrodynamics in 3D: magneto-electric effects
  - (A choice of some) consequences
- **3D topological semimetals**
  - Recap of topological properties: anomalous Hall effect
  - Modified electrodynamics
  - Non-reciprocity of optical and thermal properties



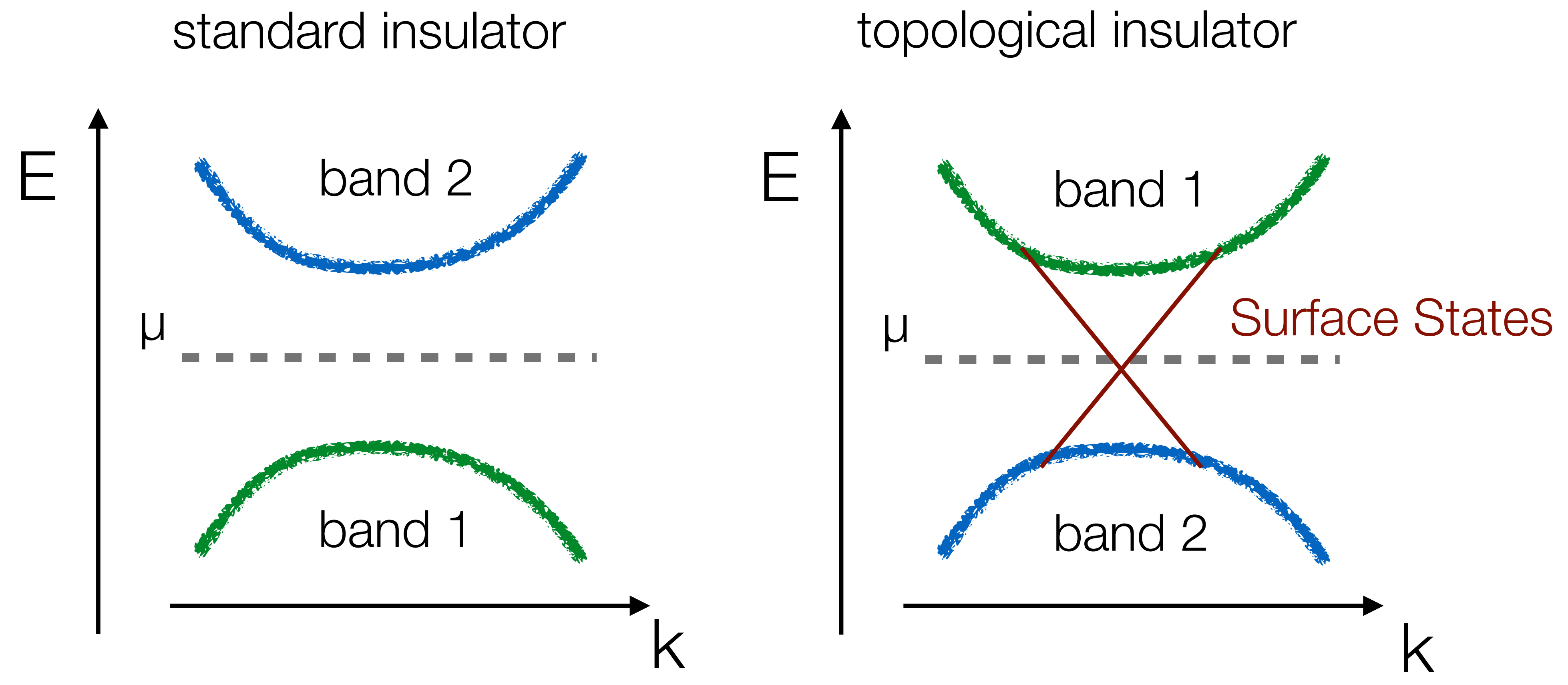



# Topological Insulators ↔ Thermoelectricity ?

**Topological Insulators** : band inversion by spin-orbit

1. large spin-orbit : materials with heavy atoms
2. gap comparable with spin-orbit : small gap semiconductors




**Chemical potential in the gap**

- same materials, different reasons ? ...
- different range of parameters

**Heavily doped**

**Thermoelectric materials :**

1. low phonon thermal conductivity : materials with heavy atoms
2. large electronic power factor $\sigma S^2$ : small gap semiconductors

# Spin-Orbit Coupling

- «**relativistic correction**»

$$-i\hbar\,\gamma^\mu(\partial_\mu + ieA_\mu)\psi = 0$$

first relativistic correction to Schrödinger

$$\left(\left[\frac{p^2}{2m} + eV\right] + \frac{e\hbar}{4m^2c^2}\,\sigma.(\nabla V \times p)\right)\psi = E\psi$$

Spin-Orbit

- favored by **heavy atoms** (high Z) : $V(r) = \frac{Ze}{4\pi\epsilon r}$ $\quad r_B \sim 1/Z^3$ $\quad \nabla V \sim Z$

$$\lambda_{SO} \sim Z^4$$

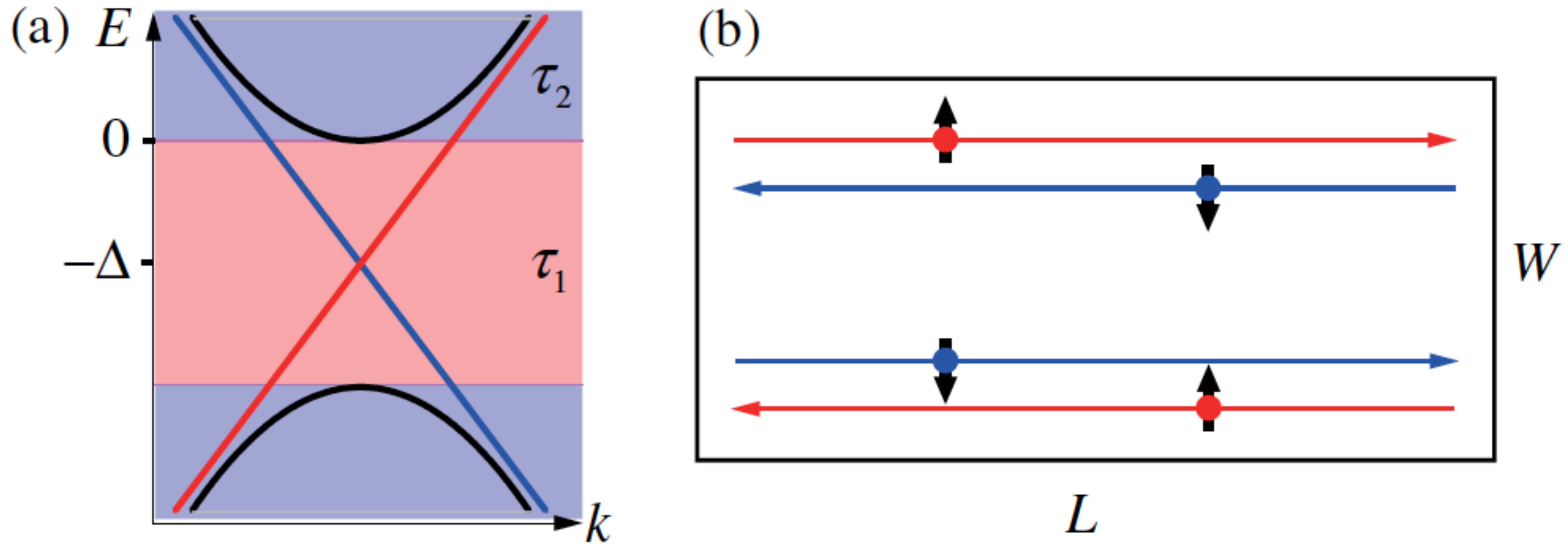
- lifts **spin degeneracy** of bands
- preserves **time reversal symmetry**

# Enhanced Thermoelectric Performance and Anomalous Seebeck Effects in Topological Insulators

Yong Xu,[1,2] Zhongxue Gan,[2] and Shou-Cheng Zhang[1,2,*]
[1]*Department of Physics, McCullough Building, Stanford University, Stanford, California 94305-4045, USA*
[2]*ENN Intelligent Energy Group, ENN Science Park, Langfang 065001, China*
(Received 5 February 2014; published 2 June 2014)

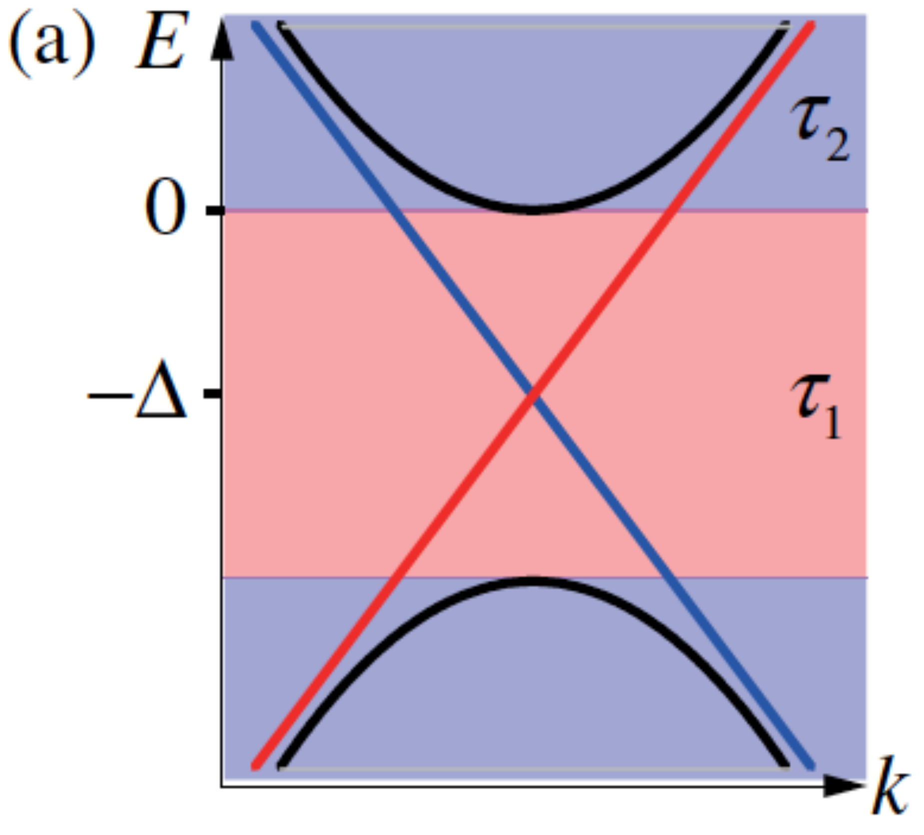
Improving the thermoelectric figure of merit $zT$ is one of the greatest challenges in material science. The recent discovery of topological insulators (TIs) offers new promise in this prospect. In this work, we demonstrate theoretically that $zT$ is strongly size dependent in TIs, and the size parameter can be tuned to enhance $zT$ to be significantly greater than 1. Furthermore, we show that the lifetime of the edge states in TIs is strongly energy dependent, leading to large and anomalous Seebeck effects with an opposite sign to the Hall effect. These striking properties make TIs a promising material for thermoelectric science and technology.

Xu et al. PRL, 112 (2014)




- 2D topological insulator (for simplicity)
- bulk / edge states contribution to transport
- two scattering times for edges states $\tau_1$ and $\tau_2$

Xu et al. PRL, 112 (2014)




- size dependance of figure of merit (bulk / edge contribution)

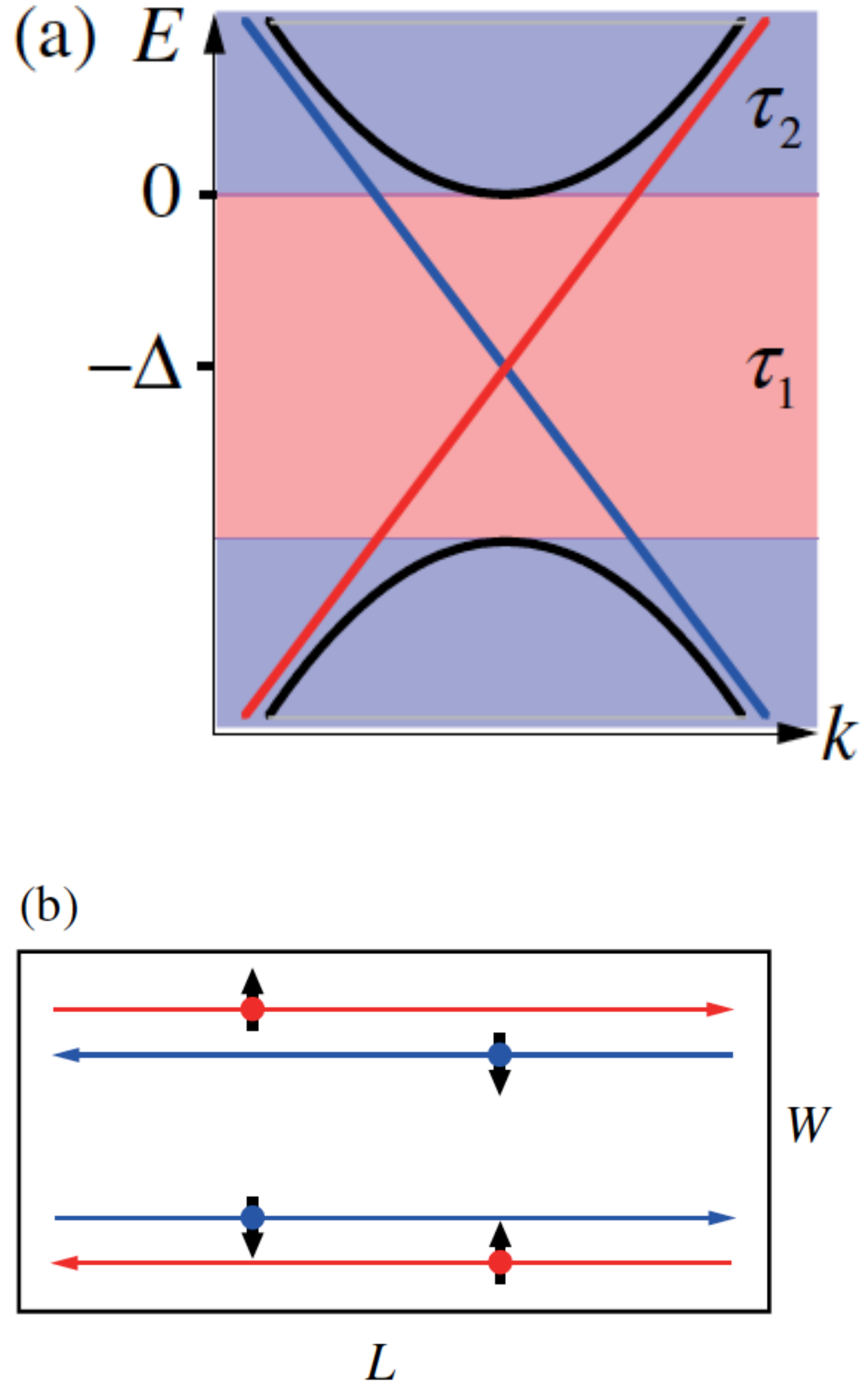
$$zT = \frac{GS^2T}{K}$$

- enhancement of Seebeck coefficient

$$S = -\frac{\pi^2 k_B^2 T}{3e} \frac{\partial \ln[\bar{\mathcal{T}}(E)]}{\partial E}\bigg|_{E=E_F}$$

usually increased by variations of density (nb of modes),
here : variation of scattering time / mean free path λ(E)

Xu et al. PRL, 112 (2014)




- size dependance of figure of merit (bulk / edge contribution)
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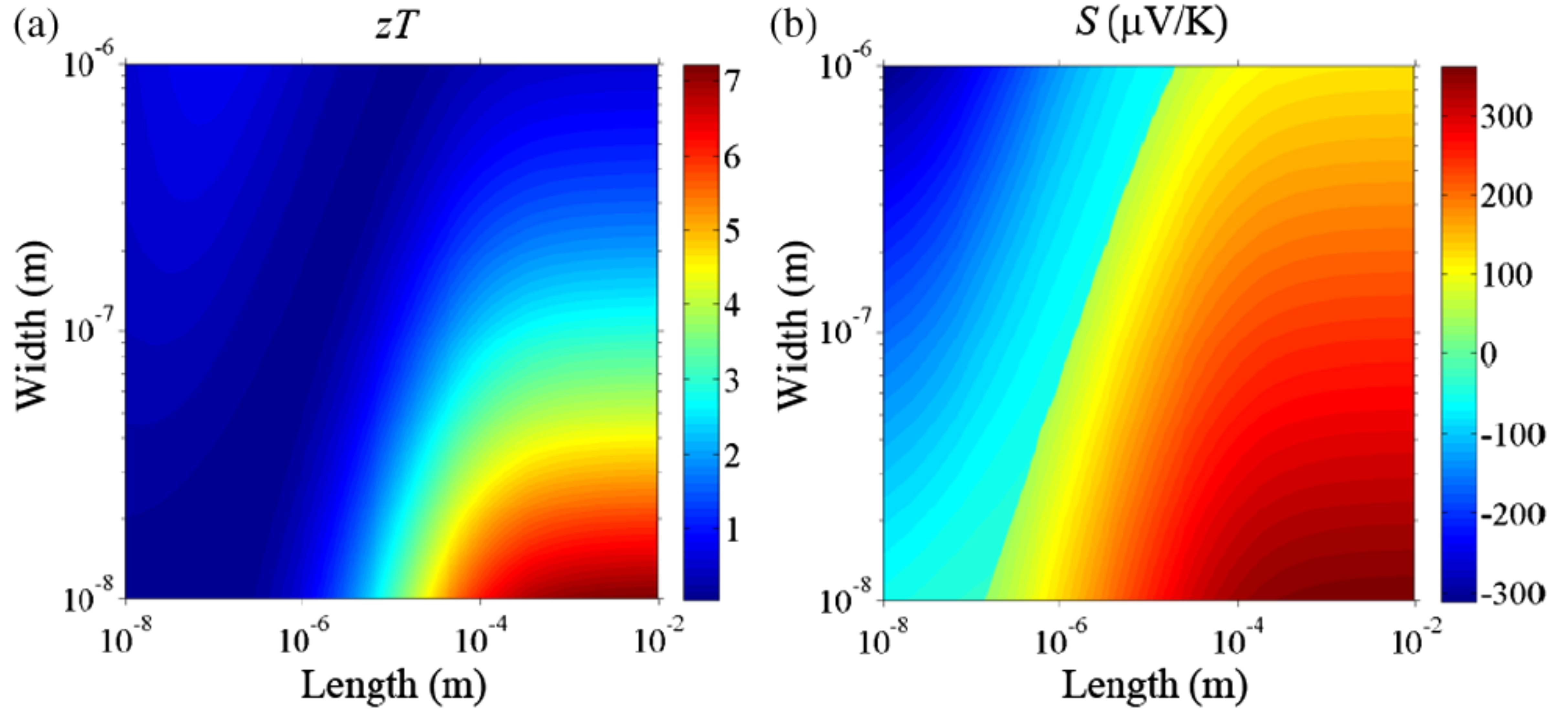


FIG. 4 (color online). The size dependence of (a) $zT$ and (b) $S$ for the 2D TI fluorinated stanene at 300 K.