Topological order and magnetism

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Contents

1 Introduction

The definition, description of properties and classification of (topological) spin liquids has recently made great progress, after many understandable trial and error in the first years of this recent domain. It seems easy to get lost in the twists and turns of *string networks*, *cohomology groups*, *anyons* and other *fractional excitations*, in the distinction between *symmetry enhanced* or *protected* phases, all this submitted to various gauge transformations, allowing to introduce *projective symmetry* and *invariance gauge groups*. This is due to the wealth of mathematical theories related to this notion of topological phase. This is without considering the collection of models supposed to explain all this to us: the Kitaev model, the toric code (also by Kitaev), the Haldane chain, the Rokhsar and Kivelson point and other dimer models... In this lecture, I try to introduce all this vocabulary, without entering too much into the details, but giving tools to go further if you need, and precise enough definitions and intuition (in the good sense). Then, we will focus on the $\mathbb{Z}_2$ spin liquids.

This is the first version (before the lecture) than I plan to amend after it. Thanks in advance to all those wo want to make comments.

1.1 Spin models

First of all, we are concerned with topological phases of spin models describing particles on a lattice. The lattice can represent a crystal (particules are electrons hopping on the sites), or an optical lattice (particules are fermionic or bosonic atoms)[7]. The interactions can be of many types. A quite general two-spin interaction Hamiltonian is:

$$H = \sum_{i,j} \left(J^z_{ij} S^z_i S^z_j + J^x_{ij} S^x_i S^x_j + J^y_{ij} S^y_i S^y_j + \mathbf{D}_{ij} \cdot (\mathbf{S}_i \wedge \mathbf{S}_j) \right) - \sum_i \mathbf{h}_i \cdot \mathbf{S}_i, \tag{1}$$

where (S^x_i, S^y_i, S^z_i) are the three components of the spin operator $\mathbf{S}_i$ on site i. The total spin S on a site is a half integer ($S = 1/2, 1, ...$ such that $\mathbf{S}^2_i = S(S+1)$). h_i is a local magnetic field. $S \to \infty$ corresponds to the classical limit where spins operators are replaced by three dimensional vectors on the $\mathcal{S}^2$ sphere[1].

In the single orbital Hubbard model with real t and Coulomb interaction U, the large-U limit leads to the simple Heisenberg model. But in more evolved and realistic models, multi-orbitals require to take into account crystal fields and spin-orbit, resulting in a whole zoo of possible spin interactions. We have illustrated this here with an anisotropic interaction (J^x_{ij}, J^y_{ij}, J^z_{ij}), together with a Dzyaloshinskii-Moriya (DM) interaction (the vector $\mathbf{D}_{ij}$), where the final spin operators are in fact pseudo-spin. DM results from the absence of some symmetries according to rules presented in [33], and often implies frustration[42] A special case of anisotropy is the Heisenberg-Kitaev interaction, where the dominant term among J^z_{ij}, J^x_{ij} and J^y_{ij} depends on the link direction, on a lattice where links can be divided in three

[1] To better understand the classical limit, think that fully polarized spin states form an overcomplete basis for any finite S, but that their scalar product tends to zero when S increases. This is formalized in the coherent spin state representation.

equivalent sets. The limit of only only one non-zero interaction on each of the three types of links is the Kitaev model[18], famous for having an exact ground state formulation in term of Majorana fermions and anyonic excitations (see later).

Other interactions are still possible when $S > 1/2$: we can have spin anisotropies as $(S_i^z)^2$, nematic interactions as $(\mathbf{S}_i \cdot \mathbf{S}_j)^2$, leading to many new phases as studied for example in [41]. More than two-particle interactions can also exist whatever the spin, as for example $(\mathbf{S}_i \cdot \mathbf{S}_j)(\mathbf{S}_k \cdot \mathbf{S}_l)$ (multi-spin exchange interactions), that have been studied a lot on the triangular lattice[32, 10].

1.2 Phases of spin models

As strongly interacting models, the set of eigenstates of spin models is not formed by tensor product of single particle ground states, nor continuously connected to such states (this is only true for non or weakly-interacting systems). The most well-known such non-interacting system is graphene, belonging to the large family of topological insulators. However, spin models can be related to topological insulators via emergent quasi-particles and emergent models.

The most interesting spin models (from our point of view) are the frustrated ones, where elementary terms of the Hamiltonian are in competition. Widely used examples are the classical antiferromagnetic triangle or the $J_1 - J_2$ square (with Ising spins or any other type of spins as XY or Heisenberg). In quantum models, competition between interactions favor superposition of states expressed in the local basis. As a trivial (unfrustrated) example of superposition, the ferromagnetic $S = 1/2$ dimer (i.e. two sites labelled 1 and 2) has unentangled ground states that are all the triplet states

$$|\psi_F(a,b)\rangle = \frac{1}{\sqrt{|a|^2+|b|^2}}(a\,|\uparrow\rangle_1 + b\,|\downarrow\rangle_1) \otimes (a\,|\uparrow\rangle_2 + b\,|\downarrow\rangle_2), \tag{2}$$

with a and b complex, while the antiferromagnetic ground state is the unique singlet state:

$$|\psi_{AF}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{3}$$

Frustration leads to more or less exotic quantum ground states as compared with ferromagnetic or Néel orders (see top of Fig. 1):

- Other types of long-range spin orders are simple generalization of ferromagnetic or Néel orders to more sublattices. They can also be obtained in classical spin models and the **linear spin wave approximation** is a nice way to introduce quantum fluctuations. Various symmetry can be broken in such ground states, that are restored via a phase trnasition either at zero (as generally expected for continuous symmetries in dimension $d \leq 2$ according to the Mermin-Wagner theorem) or at finite temperature (as the well known Ising transition). Notably, continous spins (for example Heisenberg classical spins) can give rise to finite temperature phase transition even in 2 spatial dimensions when they order in such a way that discrete lattice symmetries are broken, or the time-reversal symmetry (as for example the cuboc2 and octahedral orders of Fig. 1).

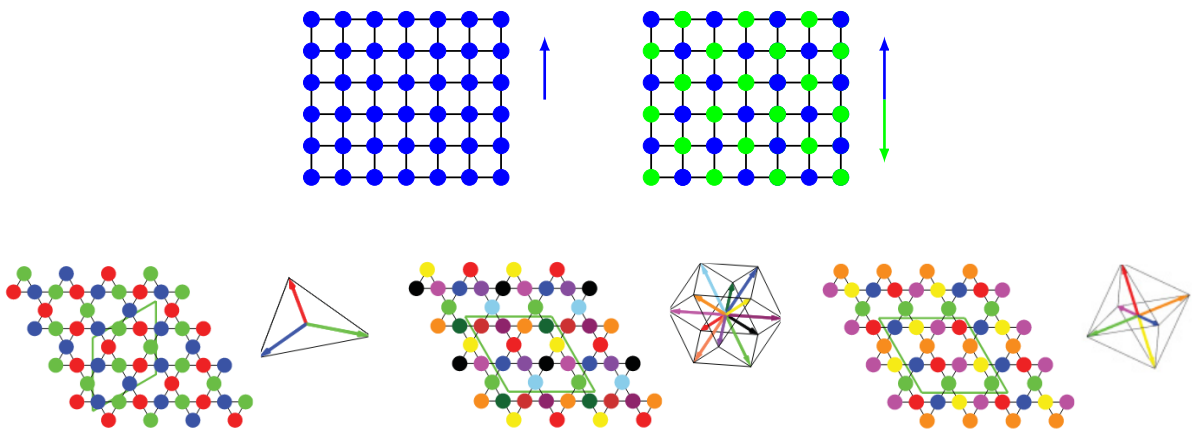


Figure 1: Top: ferromagnetic and antiferromagnetic orders on the square lattice. Bottom: examples of exotic long-range magnetic order on the kagome lattice. From left to right: the $\sqrt{3} \times \sqrt{3}$, the cuboc2 and the octahedral order.

In these spin orders that break spin rotational symmetry, zero-energy modes are present: the Goldstone modes. On the finite size spectrum of quantum models (obtained by exact diagonalization) they induce a **tower of state** collapsing as $1/N$ with N the number of sites[29].

- Other types of long-range orders are not related to the spin orientation, but to more complex order parameters, implying more than 2-spin correlations. For example, nematic (or quadrupolar) spin orders[41] are characterized by 4-spin correlations, valence bond crystals (VBC) by dimer-dimer correlations.
  VBC have been proposed as ground states of the kagome antiferromagnet[43] (Fig. 2, left), even if other phase are now expected. The ground state on the Shuriken lattice[3] has been determined to be a pinwheel VBC (Fig. 2, middle). Models have also been especially designed as two-dimensional generalizations of the Majumdar-Gosh $S = 1/2$ spin chain (where the two ground states are the singlet-dimer coverings, breaking the translational symmetry) to present VBC ground states[13].
- Finally, the remaining phases do not break any symmetry. They divide into gapless (or algebraic) and gapped phases. Among the gapped phases, we distinguish symmetry enhanced (or topological) spin liquids from symmetry protected phases (or trivial, or cooperative paramagnets) according to their entanglement properties (long or short range). We are now going to precise this distinction in the next section.

A useful remark on these various phases is that they distinguish themselves by their low temperature spectrum (see Fig. 3).

- A trivial paramagnet will have a unique gapped ground state (left).
- A state breaking some symmetry will become degenerate in the thermodynamic limit, revealing the various way to break it: a finite degeneracy with a gap for a discrete symmetry (middle), against an infinity of state and gapless modes for a continuous symmetry (due to Goldstone modes) (right).

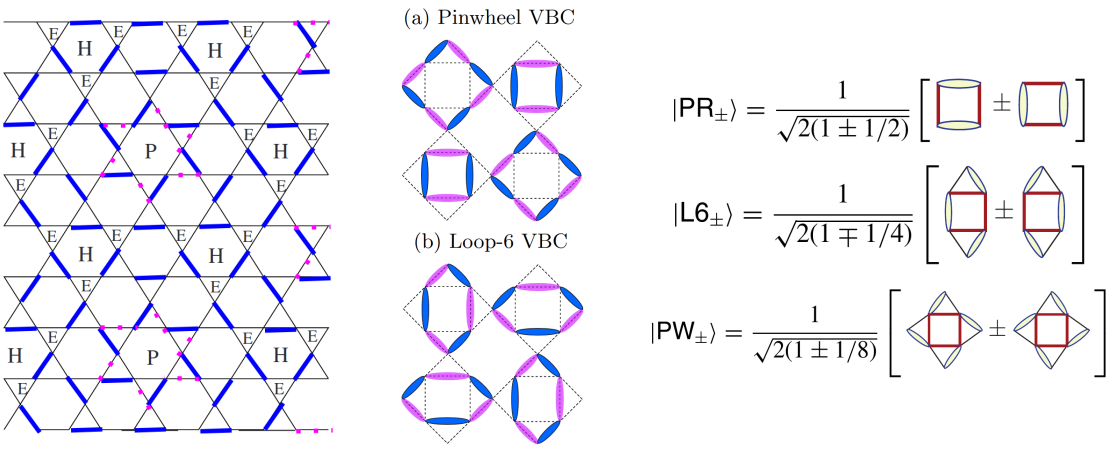


Figure 2: Example of valence bond crystals on the kagome (left, Singh and Huse 2008[43]) and Shuriken (middle, Astrakhantsev et al 2021[3]) lattices. Pinwheel and lopp-6 VBC are two concurrent patterns. Blue and magenta links represent strong dimer (meaning here singlet) density, that breaks the lattice translational symmetry. Right: pictorial representation of the wave function[3].

- A gapless state breaking no symmetry is a critical state (gapless spin liquid) (right).
- The only remaining cases are gapped degenerate phases (depending on the topology: topological degeneracy) breaking no symmetry: these are the topological spin liquids (middle).

This is related to the **Lieb-Schultz-Mattis-Hastings theorem**[37, 36, 15], stating that a system with a half-odd integer spin in the unit cell, with short range interactions, global $U(1)$ symmetry, with periodic boundary conditions in a direction and translational invariance in the same direction cannot have a gap and a unique ground state (in other words, it cannot be a SPT state).

Note that this theorem does not forbid the existence of topological spin liquids on lattices with an integer spin per unit cell. For example, the toric code on the honeycomb lattice (see

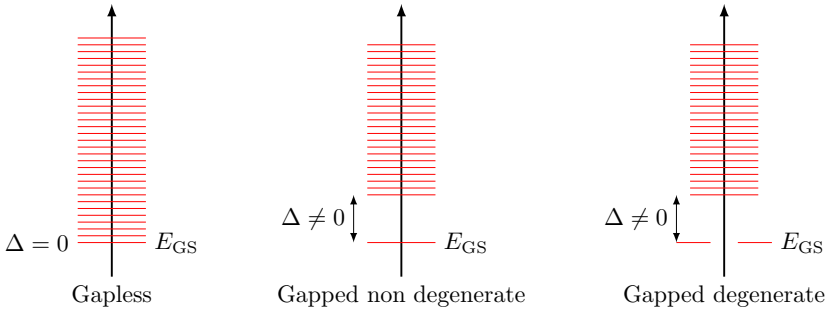


Figure 3: The three possible types of spectrum.

Sec. 5.2) is a well know example of $\mathbb{Z}_2$ spin liquid.

2 Classifications of gapped phases of matter

In this section, we have no energetic configuration, but only consider physical states (eventual ground states of future Hamiltonians) on a lattice. The classes obtained only depend on the Hilbert space. A symmetry group G can act on these states, and the action of these symmetries will be considered in a second time.

2.1 Without any symmetry

We first consider a model without taking care of the possible symmetries of states (like spin-rotational symmetry, time-reversal, lattice symmetries)... We just have a Hilbert space and a lattice. What are the possible phases in such an apparently boring system ? I follow here the line of Chen at al[9, 8] to answer this question.

States can be deformed by **local unitary** (LU) transformations. Doing so, we identify equivalence classes of states. The trivial class contains all **short-range entangled** states (SRE). Among them are the direct product states of local configurations.

In contrast, all the other classes are topological orders, possessing **long-range entanglement** (LRE). Note that several classes of LRE can be obtained without any symmetry, and without having defined any Hamiltonian.

2.1.1 Tensor categories classify long-range entangled states

LRE states can be classified using **unitary modular tensor categories**, with the milestone article of Levin and Wen[25]. Even if we do not enter the details of this mathematical theory, it is useful to be aware of the relation between it and the existence of fractional excitations of several types and gauge symmetries.

For each category, a representative (renormalization ground fixed-point) Hamiltonian (a string-net model) can be constructed, acting in a Hilbert space of string networks that we now define. This space is generated by orthogonal string configurations: wave functions specified by a value between 0 (no string) and N (the number of string types) for each oriented lattice link. The strings have to verify branching rules: only some sets of strings are allowed to meet at a vertex. The ground state in this space is further constrained by a 6-index object F^{ijk}_{lmn}, where i, j, k, l, m and n are string types and the quantum dimensions: a set of positive integers $d_{i=1..N}$. F and d relate coefficients of all allowed string configurations in the ground state.

Now, imagine that we break a string of type a by inserting the trivial string 0. We break the fusion rules and are out of the initially allowed Hilbert space. We have created two fractional particles, called anyons in 2 dimensions. They will be discussed in more details later on.

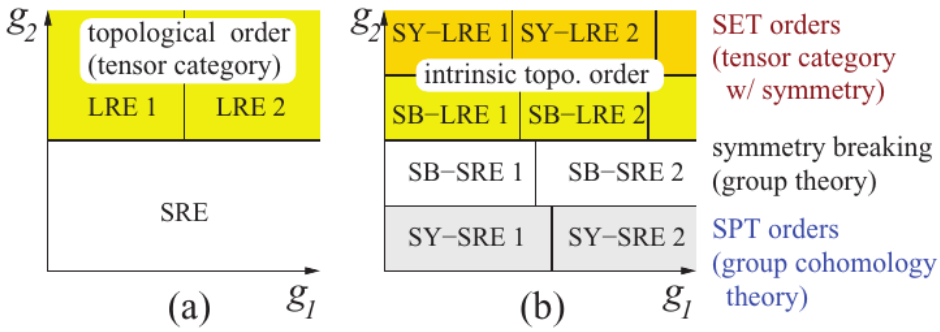


Figure 4: From [8]. States related by LU transformation are equivalent and allow to define different classes or phases (left). When some symmetries are imposed, subclasses appear (right).

2.2 With symmetries

We now define a symmetry group G. Each class of the section 2.1 (SRE and LRE phases) now divides into subclasses according to the respect or not of these symmetries. Group theory allows to label them with subgroups of G (preserved symmetries). But this labeling is not enough, as the newly allowed LU transformations between equivalent states have to respect the symmetries. Thus, different phases can have the same symmetry and the same type of short or long-range entanglement (they were equivalent before introducing the symmetries) but now belong to different classes. In SRE as in LRE, the subclasses labeled by G itself (phases breaking no symmetry) have a special role: those from SRE and LRE are dubbed respectively **symmetry protected topological** (SPT) and **symmetry enriched topological** (SET) phases.

All the phases introduced in this two last sections are pictorially represented on Fig. 4.

2.2.1 Symmetry protected topological (SPT) phases

SPT phases can be labeled by $d+1$ **group cohomology classes**[8, 28] (definition in App. A of [5]). The most well know examples of SPT phase are

- the Haldane $S = 1$ chain, protected by $SO(3)$ spin rotation symmetry: it has fractional edge mode, despite no ground state degeneracy and no long-range entanglement.
- the topological insulators[14] or topological superconductors, that are classified in ten discrete symmetry classes corresponding to the ten Altland–Zirnbauer classes of random matrices (1997)[1] (so-called ten-fold way, or K-theory), and band theory with Chern numbers.

Here are examples of SPT phases in 2 dimensional magnetic models (also called cooperative or trivial paramagnets[21]):

- The $S = 1/2$ Shastry-Sutherland model with two Heisenberg exchanges J and J' (see Fig. 5) has a unique ground state of dimers on the J' links when $J \gtrsim 0.675J$[44,

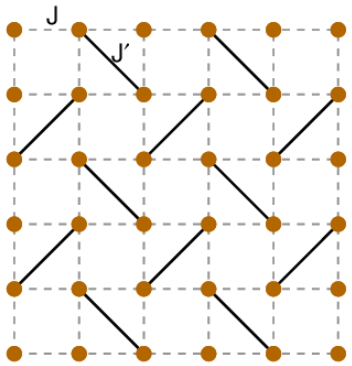


Figure 5: The Shastry Sutherland model, with two Heisenberg exchange parameters J and J'. For $J \gtrsim 0.675J'$, the ground state is a tensor product of dimers on the J' links.

11]. From the magnon point of view, this is a bosonic topological insulator with topologically protected chiral edge modes of triplon excitations[27],

- The transverse Ising model on the kagome lattice, shown to be smoothly connected to the high field state as soon as the field is non-zero[35].
- In [23], two simple models with an Ising symmetry are constructed and studied in great details, one being a trivial and the other a non-trivial SPT phase.

2.2.2 Symmetry enriched topological (SET) phases

They respect all the symmetries of the model, and possess long-range entanglement. As before introducing symmetries, the insight of tensor categories remains valid, but a supplementary classification has to be determined inside each category, implying **projective symmetry classification**.

These phases are the focus of this lecture, and still more specifically, topological $\mathbb{Z}_2$ spin liquids. Topological ($\mathbb{Z}_2$ or not) spin liquids (or equivalently gapped spin liquids) are before all spin liquids[40, 21, 4], that are roughly defined as gapped states breaking no symmetry down to zero temperature. But this definition does not allow to distinguish the SPT phases as the two examples given in Sec. 2.2.1. More precise and equivalent definitions are, as for any SET phase::

- states with non-local fractional excitations (partons), that are anyons in 2d (Sec. 3)
- states with a non-zero topological entanglement entropy γ (Sec. 4)
- states with a topological degeneracy.

We focus here on gapped $\mathbb{Z}_2$ spin liquids[12]. Several Mott insulators are candidates: the $J_1 - J_2$ square lattice near $J_2 = J_1/2$, the $J_1 - J_2$ triangular lattice near $J_2 = J_1/10$, the kagome antiferromagnet (but more probably a gapless spin liquid).

We will give the example of two very simple models of $\mathbb{Z}_2$ spin liquids, before entering in the more precise SET classification for more realistic spin models. Before that, the definitions of anyons and topological entanglement entropy are given.

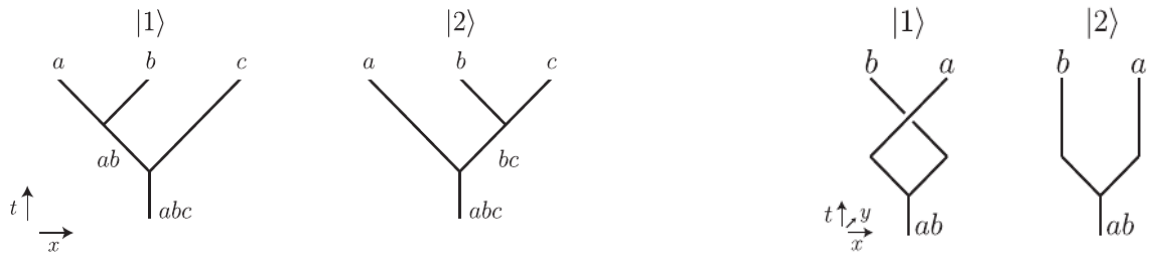


Figure 6: Graphical representation of the F (left) and R (right) symbols defining anyons, from [16].

3 Fractional quasi-particles, anyons

The first question that arise is the precise definition of quasiparticles (QP) and anyons and of their properties in a gapped system with a short-range interactions Hamiltonian H. We limit ourselves here to abelian fractional quasiparticles theory and use the definitions given in[16].

The first data that characterize a model is the set of QP types $\{a, b, ...\}$. In 2 dimensions, the QP are called anyons, and can be different from bosons or fermions. In 3 dimensions, only bosons and fermions are allowed. Moreover, in 2 dimensions, their mutual statistics can be non trivial, while they are triial in three dimensions.

We consider the set of Hamiltonians $H+V$ on the infinite plane (this condition is required to have the possibility of a unique QP, sending the other to infinity), where V is a local perturbation. The ground states obtained for all possible V form equivalence classes up to local unitary transformations. We get as many equivalence classes as QP, including the trivial one 1 corresponding to the class of the ground state. V can be seen as a trapping potential for a QP. Another denomination used is **superselection sectors**, to refer to these equivalence classes of states, with the trivial superselection sector for the vacuum (no QP).

Secondly, the **fusion rules** indicate what QP type c is obtained by merging a pair of QP a and b. For abelian QP, the fusion rules are simply of type $a \times b = ab = c$.

In two dimensions, the **F-symbol** inform on the way an anyon abc splits in three a, b and c, depending on the steps (first ab and c or a and bc), as pictorially described in Fig. 6 left and lastly, the **R-symbol** indicates the exchange statistics of mutual anyons (Fig. 6 right):

$$|1\rangle = F(a,b,c)\,|2\rangle\,, \qquad\qquad |1\rangle = R(a,b)\,|2\rangle\,. \tag{4}$$

Some examples of anyons:

- The best studied example (both theoretically and experimentally) is the Laughlin state in the fractional quantum Hall system at the filling $\nu = 1/3$, with abelian anyons with exchange phase $\phi = \pi/3$ and electric charge $e = \pm 1/3$.
- The more detailed example of non-abelian anyons occurs in the fractional quantum Hall system at the filling $\nu = 5/2$[34].

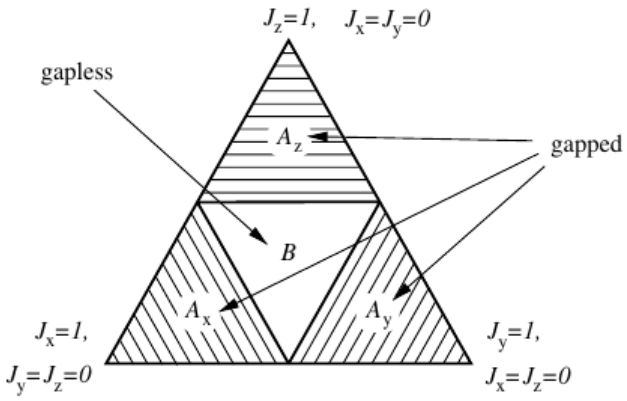


Figure 7: Phase diagram of the Kitaev model Eq. (5).

- Kitaev model[18] defined on the honeycomb lattice, with exchange that depend on the three possible link orientation and couple the x, y or z components.

$$H = -J_x \sum_{x\text{-links}} S_j^x S_k^x - J_y \sum_{y\text{-links}} S_j^y S_k^y - J_z \sum_{z\text{-links}} S_j^z S_k^z. \tag{5}$$

  For each hexagonal plaquette $p = 123456$, an operator W_p is found that commutes with the Hamiltonian, and with other such operators. Thus, solving H reduces to solving its reduction to a each eigenspaces of W_p.
  This more simple problem itself reduces to solving a model on non-interacting Majorana fermions, giving the phase diagram of Fig. 7
  The gapped phase is qualitatively similar to the toric code phase, obtained via a mapping from perturbation theory in the case $J_z \gg J_x, j_y$ (or permutations).
  On the other side, the gapless phase acquires a gap in the presence of a magnetic field, and a non trivial spectral Chern number $\nu = \pm 1$ (depending on the direction of the magneitc field), associated to the existence of gapless edge modes.

4 Topological entanglement entropy

Let a system in a wave function $|\Psi\rangle$. Its density matrix is simply $\rho = |\Psi\rangle\langle\Psi|$ and we have a pure state (entropy $\mathrm{Tr}\rho\ln\rho = 0$).

The von Neumann entanglement entropy between two regions A and $B = \overline{A}$ (Fig. 8) of smooth boundaries is:

$$S(A) = -\mathrm{Tr}_A \rho_A \ln \rho_A, \qquad \rho_A = \mathrm{Tr}_B \rho, \tag{6}$$

For most ground states (and specifically for gapped ground states), the dominant behavior of $S(A)$ follow the **area law**:

$$S(A) \sim s_0 L, \tag{7}$$

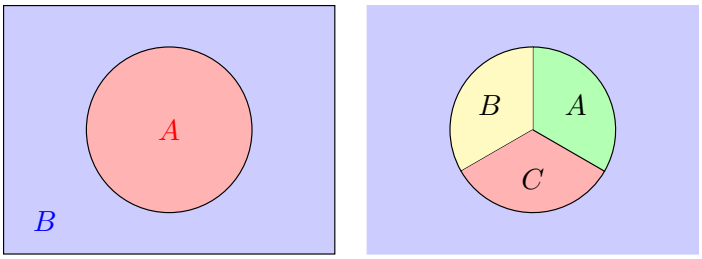


Figure 8: Left: Subsystem A of a larger subsystem AB.
Right: the topological entanglement entropy γ can be calculated by linear combinations of the entanglement entropies of subsystems A, B, C, AB, AC, BC and ABC.

with L the length of the boundary between A and B and s_0 is a non universal term. The term *area* comes from the dimension 3, where the dominant term is $s_0 A$, with A the area of the boundary of the sybsystem. In 2 dimension, this is a perimeter law, however, the use is to keep the 3d denomination.

The next subdominant term is noted γ and called the **topological entanglement entropy**. It takes only quantized values and depends on the topological phase:

$$S(A) = s_0 L - \gamma + o(1). \tag{8}$$

The negative sign indicates that the state has a long-range entanglement that provides more information (less entropy) than in a non-topological phase.

γ enlights the fact that the topological properties of a phase is independent on its Hamiltonian or on its excitations (dynamical properties), but only depend on the wave function. In two paper published simultaneously[19, 24], a clever way to determine γ despite its subdominant character is explained: it consist in a combination of the entanglement entropy of several subsystems (see Fig. 8). Consider

$$S_{\text{topo}} = S_{ABC} - S_{AB} - S_{CB} - S_{CA} + S_A + S_B + S_C, \tag{9}$$

the term coming from the perimeters cancels, and only $-\gamma$ remains.

The topological entanglement entropy can be related to the number and the nature of the anyons, or equivalently, to the number and nature of strings in the associated string-net model:

$$\gamma = \ln D^2, \tag{10}$$

with D the so-called total quantum dimension of the topological order. It is 1 for a topologically trivial phase, and verify the two following equalities, to be carefully distinguished:

$$D = \sum_i d_i^2 = \sqrt{\sum_\alpha d_\alpha^2}, \tag{11}$$

The first sum concerns string net models (for abelian theories, it is simply the number of string types plus one), while the second sum is for anyon theories (for abelian theories, it is the number of superselection sectors).

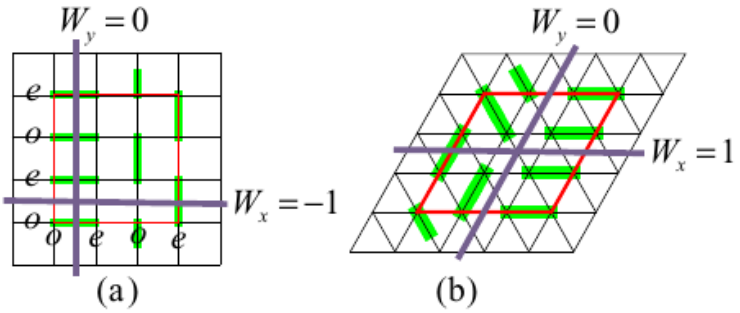


Figure 9: Definition of winding numbers W_x and W_y on bipartite and non-bipartite lattices (from [22]).

5 Topological $\mathbb{Z}_2$ spin liquids in toy models

5.1 Example 1: the resonating valence bond phase

The resonating valence bond (RVB) phase was introduced in 1976 by Anderson[2] and proposed to describe the ground state on the triangular antiferromagnetic lattice (later shown to be wrong). States with exactly one dimer per site towards one of the nearest neighbors represent a tensor product of singlets. They do not generate the whole spin Hilbert space (missing for example magnetized states, and not even the $S = 0$ sector (however, they form an overcomplete basis in this sector when long-range dimers are allowed). Moreover, they are not orthogonal. However, the approximation of an orthonormal basis of dimer states allows to define tractable models of dimers with a rich physics.

The Rokhsar and Kivelson model Only configurations with one (neither zero nor more) dimer emanating from each site are allowed (closed packed dimer configurations) and form an orthogonal basis. The Hamiltonian is:

$$H_{\mathrm{RK}} = \sum_{plaquettes} \left[-J \left(|\square\rangle\langle\square| + |\square\rangle\langle\square| \right) + V \left(|\square\rangle\langle\square| + |\square\rangle\langle\square| \right) \right] . \tag{12}$$

The model has initially been defined on a square lattice, but it was then extended to other lattices. The first term in the Hamiltonian is a kinetic term (or resonant, at the origin of the term RVB), and the second a potential term.

Winding numbers The configuration space on a torus divides into winding sectors, labeled by two winding numbers W_x and W_y. They are obtained on a bipartite lattice by assigning a positive direction to links from sublattice A to B on the square lattice, and summing the contributions of dimers on a reference loop in the x and y directions (see Fig. 9). We verify that any local move (flip of dimers long a loop as in Fig. 10) conserve these numbers,

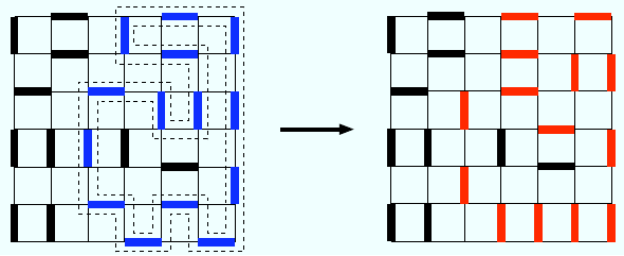


Figure 10: Local operators consist in flip of dimers around loops (from [30]: occupied and empty links are exchanged.).

that are in $\mathbb{Z}$. This implies that the Hamiltonian does not mix topological sectors: winding numbers are good quantum numbers.

On a non-bipartite lattice, only the parity of the number of dimers crossing the reference loops has a meaning, and W_x, $W_y \in \mathbf{Z}_2$.

Phase diagram The $J = V$ point (called the RK point) is soluble, as the Hamilonian rewrites as a sum of projectors:

$$H_{\mathrm{RK}} = \sum_{plaquettes} \left(|\square\rangle - |\square\rangle \right) \left(\langle\square| - \langle\square| \right) . \tag{13}$$

The RVB state

$$|RVB\rangle = \frac{1}{\sqrt{N_c}} \sum_c |c\rangle \, , \tag{14}$$

is the sum of all dimer configurations in a given winding sector (W_x, W_y). It is an eigenstate of zero energy and consequently, it is a ground state. On the square lattice, we obtain an extensive number of ground states (one in each winding sector), but on the triangular lattice, we get a degeneracy of 4, typical of a $\mathbf{Z}_2$ spin liquid. Another difference between these two lattices is the nature of the dimer-dimer correlations in the RVB state: they are algebraic on the square lattice, against exponentially decreasing on the triangular (same calculation and result as for $T = \infty$).

On the square lattice, the RK point is thus a critical ground state, unstable to any perturbation (see Fig. 11). It separates a staggered phase (with no flippable plaquettes) from a columnar or plaquette phase for a negative strong potential term. On a number of bipartite lattices in $d \geq 3$, including the cubic lattice, the RK point is part of a Coulomb phase (or $U(1)$ RVB).

In contrast ,on non-bipartite lattices in two and higher dimensions, including the 2d triangular[31] and 3d fcc, the RK point is part of a $\mathbb{Z}_2$ gapped RVB liquid phase[31], separating as previously a columnar and staggered phase with still an intermediate VBC phase. This time, the $V = J$ point is not critical, and the transition is first order (the RVB state is gapped).

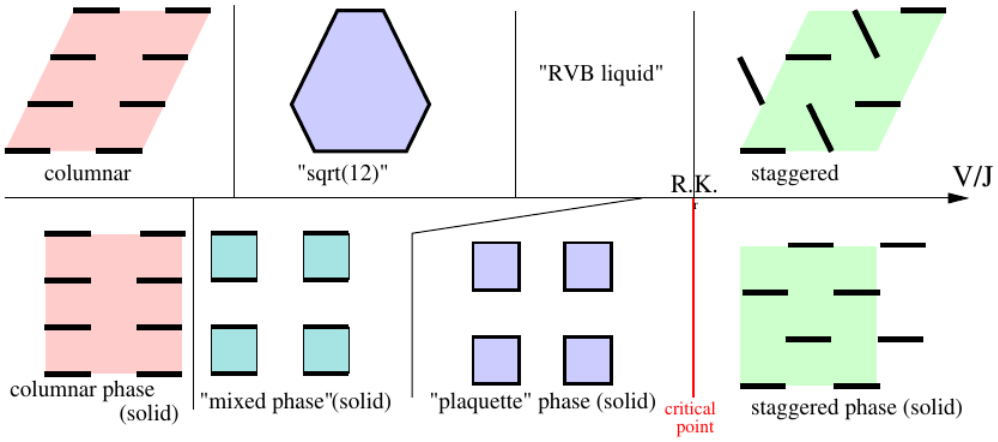


Figure 11: Phase diagram of the RK model (12) on non-bipartite (top) and bipartite lattices (bottom)[45]).

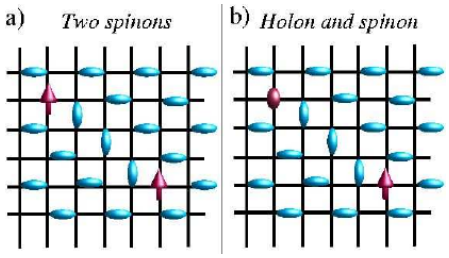


Figure 12: Dimer configurations with fractional excitations spinons and holons in a QDM model (from [6]).

Finally, for a number of 2d bipartite lattices, including the square and honeycomb, the RK point is a special critical point separating different crystalline phases. A nice review on quantum dimer models is here[30].

Fractional excitations We may now have to enlarge our Hilbert space (and add a term in the Hamiltonian to penalize the new states: the properties discussed before are unchanged).

We allow to break a dimer (a singlet) into a triplet. Two quasiparticles are created, with no charge and a spin 1/2. They are called **spinons** (see Fig. 12).

Still another possible excitation on non-bipartite lattices is a vison is a **vison** localized on a site i, of wave function is a sum over state in a winding sector with $\pm$ signs (see Fig. 13:

$$|V\rangle = \sum_c (-1)^{n_c} |c\rangle \,. \tag{15}$$

Note that this does not seem to enlarge the basis.

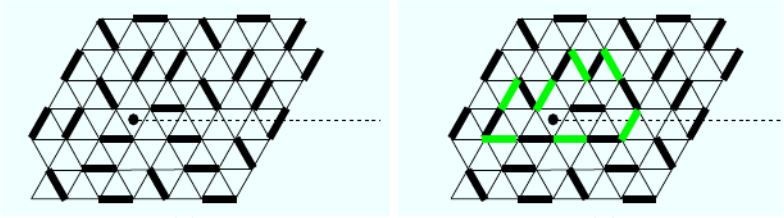


Figure 13: Left: a vison on a non bipartite lattice, with a string going from site i to infinity (from [30]. Right: moving a spinon around a vison changes the sign of the state.

We can also authorize the removal of two electrons belonging to a dimer ij. We get two sites with no dimer, creating a new type of quasiparticle with a charge but no spin, called a **holon** (see Fig. 12). But this excitation belongs to the same class as a spinon: we can see it simply because both excitations are wore by a site. If we send far apart a holon of a pair, and create a pair of spinons near the remaining holon, a local change in the configuration gives a unique spinon. This is obtained by merging a holon and a spinon on neighboring sites into a dimer. Still another way to see it is that creating and annihilating a pair of visons or holons, and annihilating them after a topologically non trivial loop move swap the dimers along the loop in both cases.

The way we enlarge the Hilbert space (with holons or spinons) is not fundamental. By the same way, it seems that the Hilbert space was not enlarged by visons, but the choice of the basis is arbitrary. We could have chosen as a basis the only four ground states, which would have require to enlarge it to have visons.

With loop operators, it is possible to separate the two quasiparticles far one from each other without any energy cost: they are said to be deconfined. When confinement occurs (through spinon condensation), it breaks the spin liquid phase and form crystalline phases like the staggered or columnar ones[39].

5.2 Example 2: the toric code

The toric code model[20] (designed by Kitaev, but be aware that there are several Kitaev models !). It consists in $S = 1/2$ spins living on the edges of a square lattice, on a torus. These model is well described and analyzed for example in [12].

For each vertex v and face p, we consider the operators (see Fig. 14):

$$A_s = \prod_{j \in \text{star } (s)} \sigma_j^x, \qquad B_p = \prod_{j \in \text{boundary } (p)} \sigma_j^z. \tag{16}$$

They commute with each other

$$[A_s, A_{s'}] = [B_p, B_{p'}] = [A_s, B_p] = 0$$

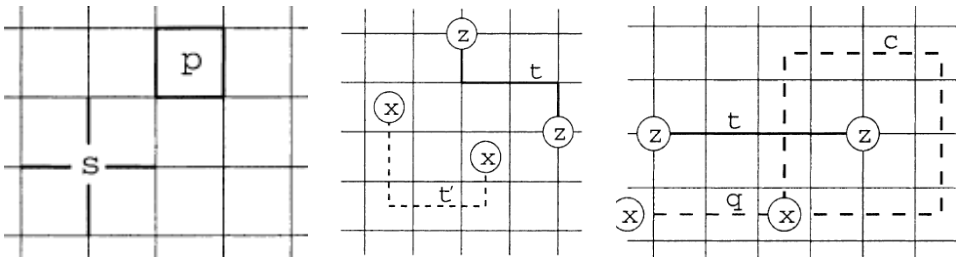


Figure 14: Left: stars and plaquettes on a toric square lattice. Right: string operators. t changes the sign of two A_s operators at z, whereas t' changes the sign of two B_p operators ar x. Pictures from [20].

and have eigenvalues ± 1 with equal multiplicity. We verify easily that the product over all stars $\prod_s A_s$ and over all plaquettes $\prod_p B_p$ are both equal to one. Each sector of fixed A_s and B_p has dimension 4.

The Hamiltonian:

$$H_{\text{tc}} = -K_e \sum_s A_s - K_m \sum_p B_p \tag{17}$$

with $K_e, K_m > 0$ thus has a gapped ground state with degeneracy 4, with $A_s = B_p = 1$ for any star s and plaquette p.

Fractional excitations Two elementary excitations e and m arise naturally:

- by applying σ_i^z on a bond i, we change the sign of A_s on two sites and create a pair of excitations (say electronic charges e). When we apply $\prod_t \sigma_i^z$ on an open string between sites of the lattice, we similarly create two e particule, at the end of the string (see Fig. 14). Applying this product on a loop around one of the two direction of the torus gives a new ground state.
- by applying σ_i^x on a bond i, we change the sign of B_p on two plaquettes and create a pair of excitations (magnetic charges m). When we apply $\prod_{t'} \sigma_i^x$ on links perpendiculars to an open string between sites of the dual lattice (centers of square), we similarly create two m particule, at the end of the string (see Fig. 14). Applying this product on a loop around one of the two direction of the torus gives a new ground state.

Two e particles can be exchanged without changing the phase of the wave function: they are bosons. It's the same for the m quasiparticles. However, the mutual statistics of electric and magnetic charges is *semionic*, as the state changes sign after one is moved around the other (see Fig. 16 right): they are mutual semions.

A third and last elementary excitation (with the trivial one 1) is the $e - m$ bound state ϵ. Due to the exchange statistics between e and m, ϵ is a fermion and changes the wave function sign when turning around a e or m particle.

Fusion rules The way two particles merge to give a new one are given by the fusion rules:

$$\begin{aligned} &e \times e = m \times m = \epsilon \times \epsilon = 1 \times 1 = 1, \\ &e \times 1 = e, \quad m \times 1 = m, \quad \epsilon \times 1 = \epsilon, \\ &e \times m = \epsilon, \quad e \times \epsilon = m, \quad m \times \epsilon = e. \end{aligned} \tag{18}$$

Ground state degeneracy Creating a pair of electric (or magnetic) charges e (or m) at a point and annihilating them after moving one around a topologically non trivial loop of the torus gives distinct but same energy states. When doing this in an eigenstate, for e, m and ϵ, we get four degenerate states as already obtained by a counting of same energy states.

Equivalent string-net model This model is an example of string-net model from Levin and Wen[25] mentioned in Sec. 2.1.1, even if the connection with string-net models is not apparent at this stage. To see it, we depart from the state with all spins up, denoted $|\uparrow\rangle$. This is clearly an eigenstate for all B_p of eigenvalues 1. Now, acting on this state with the projector on eigenstates ofA_s with eigenvalues 1:

$$|\Psi\rangle \propto \prod_s (1 + A_s) |\uparrow\rangle \tag{19}$$

conserves all $B_p = 1$ and projects on a ground state $|\Psi\rangle$. If we consider that a state with $\sigma_x = \downarrow$ is occupied by a string, and $\sigma_x = \uparrow$ by no string, then this ground state is the superposition of all closed string configurations on the square lattice, and we get a string model with $N = 1$.

Entanglement entropy We can calculate exactly this γ term in the toric code model[24], using the expression of a ground state as the superposition of all loop configurations (19). We chose as boundary a loop crossing L links and no site. We can decompose the states into $\sum_l |\Psi_l^{\text{out}}\rangle |\Psi_l^{\text{in}}\rangle$, where l runs over all string configurations on the L links of the boundary. Any configuration with an even number of links is allowed (their number is 2^{L-1}). We have written our state such that its density matrix $\sum_l |\Psi_l^{\text{out}}\rangle |\Psi_l^{\text{in}}\rangle \langle\Psi_l^{\text{out}}| \langle\Psi_l^{\text{in}}|$ is diagonal (as the $|\Psi_l^{\text{out}}\rangle |\Psi_l^{\text{in}}\rangle$ are orthogonal). The reduced density matrix is $\sum_l |\Psi_l^{\text{in}}\rangle \langle\Psi_l^{\text{in}}|$, and as all wave functions have the same weights, its entropy is $\ln(2^{L-1}) = L \ln 2 - \ln 2$. We have determined that $\gamma = \ln 2$, as expected from Eq. (11), for a string model with $d_0 = d_1 = 1$, and an anyon theory with 4 superselection sectors.

Note that any combination of the four ground states would have given the same γ.

Remark: this very simple model can be related to the so-called $\mathbb{Z}_2$ model, or $\mathbb{Z}_2$ lattice-gauge theory[24]. Its Hamiltonian is $-K_m \sum_p B_p$, and the Hilbert space only consists in $A_s = 1$ states. Only the m quasi-particles exist and are called **visons**.

Figure 15: Calculation of the entanglement entropy of the area inside the blue loop for a loop configuration (from [24]).

6 Symmetry enhanced $\mathbb{Z}_2$ spin liquids using Schwinger boson mean-field theory

In this last section, we add the constraint that the spin model possesses some symmetries and explore the consequences in a specific approach: the Schwinger boson mean-field theory, that allow to describe $\mathbb{Z}_2$ spin liquids.

After the toy models introduced previously, we would like to treat models that describe common Mott insulators. Typically, the Heisenberg model;

$$H = \sum_{\langle i,j \rangle} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j. \tag{20}$$

where spins have $S = 1/2$. This model is not soluble in the presence of frustration, and many tools have been developed to handle it:

- **Exact diagonalisations** are very interesting as they give access to the low energy spectrum with all quantum numbers (using the irreducible representations of the symmetry group). They allow to study the broken symmetries. However, they are limited to small sizes (less than 50 in the total spin 0 sector).
- **Quantum Monte-Carlo** simulations are quasi-exact in classical spin systems, but present the sign problem (statistical weight of the sampled configurations that can be negative) in the presence of frustration in quantum systems. For spin models, two largely used methods[17] are Stochastic Series Expansions (SSE) and world-line Monte-Carlo methods using for example worms algorithms to update the configurations.
- A large family consists in the **tensor network methods**. Variational states are stored under the form of for example Matrix Product States (MPS) or Projected Entangled Pair States (PEPS), are optimized to approximate the ground state using Density Matrix Renormalization Group (DMRG) for MPS, imaginary time evolution or other techniques for PEPS[38].

We now detail the Schwinger boson mean-field theory, and its fermionic analog with Abrikosov fermions.

6.1 The parton construction

We want to be able to describe fractionalized quasi-particles (or *partons*) and rewrite the Hamiltonian in terms of spinons. They are $S = 1/2$ chargeless particules, that can only be created in pairs. In non-topological phases, they are confined: they do not exist alone, as the energy to separate them is infinite. In a topological phase, we say the phase in deconfined, and spinons are free. We can chose bosonic $(b^\dagger_{i\sigma}, b_{i\sigma})$ or fermionic $(f_{i\sigma}, f^\dagger_{i\sigma})$ operators, creating or annihilating spinons of spin $\sigma =\uparrow,\downarrow$ on site i. The spin operators are

$$\begin{aligned} 2S^z_i &= b^\dagger_{i\uparrow} b_{i\uparrow} - b^\dagger_{i\downarrow} b_{i\downarrow} = n_{i\uparrow} - n_{i\downarrow} \\ S^+ &= S^x + iS^y = b^\dagger_\uparrow b_\downarrow, \\ S^- &= S^x - iS^y = b^\dagger_{i\downarrow} b_{i\uparrow}. \end{aligned}$$

This rewrite more compactly:

$$\mathbf{S}_i = \frac{1}{2}\psi^\dagger \vec{\sigma} \psi \tag{21}$$

where

$$\psi = \begin{pmatrix} b_{i\uparrow} \\ b_{i\downarrow} \end{pmatrix} \quad \text{for bosons,} \qquad \psi = \begin{pmatrix} f^\dagger_{i\uparrow} \\ f_{i\downarrow} \end{pmatrix} \quad \text{for fermions.} \tag{22}$$

For future use, we can determine all possible quadratic operators on a link ij that are invariant by a global spin rotation. These are generated for bosons by the four operators A_{ij}, $A^\dagger_{ij}$, B_{ij} and $B^\dagger_{ij}$ defined by:

$$\begin{aligned} A_{ij} &= \frac{1}{2}(b_{i\uparrow} b_{j\downarrow} - b_{i\downarrow} b_{j\uparrow}) \\ B_{ij} &= \frac{1}{2}(b^\dagger_{i\uparrow} b_{j\uparrow} + b^\dagger_{i\downarrow} b_{j\downarrow}), \end{aligned}$$

(similar operators exist for the fermions).

The Hamiltonian in terms of spinons is quartic (terms as $b^\dagger b b^\dagger b$) and can be written using A and B operators:

$$H = \sum_{\langle i,j \rangle} J_{ij} (: B^\dagger_{ij} B_{ij} : - A^\dagger_{ij} A_{ij}), \tag{23}$$

The Hilbert space of spinons is larger that the spin space. We have to respect the constraint on the spinon number per site:

$$n_\uparrow + n_\downarrow = 1. \tag{24}$$

We have a new model, with local $G = U(1)$ gauge transformations for bosons:

$$g : b_{j\sigma} \to e^{i\theta_j} b_{j\sigma}, \qquad \theta_i \in \mathbb{R}. \tag{25}$$

Actually, for fermions, the local gauge group is $G = SU(2)$:

$$g : \Psi_j \to M_i \Psi_i, \qquad \Psi_j = \begin{pmatrix} f_{\uparrow j} & f_{\downarrow j} \\ f^\dagger_{\downarrow j} & -f^\dagger_{\uparrow j} \end{pmatrix}, \quad M_j \in SU(2). \tag{26}$$

6.2 The mean-field approximation

To get a soluble quadratic Hamiltonian, we use a mean-field decoupling:

$$\begin{aligned} bbbb &= (bb - \langle bb\rangle)(bb - \langle bb\rangle) + \langle bb\rangle bb + bb\langle bb\rangle - \langle bb\rangle\langle bb\rangle. \\ &\simeq \langle bb\rangle bb + bb\langle bb\rangle - \langle bb\rangle\langle bb\rangle \end{aligned} \tag{27}$$

Applied to Eq.(23):

$$\begin{aligned} H_{\mathrm{MF}} = \sum_{\langle i,j\rangle} J_{ij} \Big(&\langle B_{ij}^\dagger\rangle B_{ij} + \langle B_{ij}\rangle B_{ij}^\dagger - \langle A_{ij}^\dagger\rangle A_{ij} - \langle A_{ij}\rangle A_{ij}^\dagger \\ &- |\langle B_{ij}\rangle|^2 + |\langle A_{ij}\rangle|^2 \Big) - \sum_i \lambda_i (n_i - 1), \end{aligned} \tag{28}$$

The last term is related to the constraint, with a Lagrange multiplier. We defined a Hamiltonian depending on complex link parameters $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$ and real site parameters λ_i. We call these parameters an ansatz.

$$H_{\mathrm{MF}} = \sum_{\langle i,j\rangle} J_{ij} \left(\mathcal{B}_{ij}^* B_{ij} + \mathcal{B}_{ij} B_{ij}^\dagger - \mathcal{A}_{ij}^* A_{ij} - \mathcal{A}_{ij} A_{ij}^\dagger - |\mathcal{B}_{ij}|^2 + |\mathcal{A}_{ij}|^2 \right) - \sum_i \lambda_i (n_i - 1). \tag{29}$$

The Hamiltonian can be written under a matricial form:

$$H_{\mathrm{MF}} = v^\dagger M v + \epsilon_0,$$

with $v^\dagger = (b_{\uparrow 1}^\dagger, \ldots, b_{\uparrow N}^\dagger, b_{\downarrow 1}, \ldots, b_{\downarrow N})$, M a $2N \times 2N$ matrix, N the number of sites and ϵ_0 a constant.

It can be solved using a Bogoliubov transformation to express the Hamiltonian in the diagonal form

$$H_{\mathrm{MF}} = \sum_{j\sigma} \epsilon_j \tilde{b}_{\sigma j}^\dagger \tilde{b}_{\sigma j} + \tilde{\epsilon}_0. \tag{30}$$

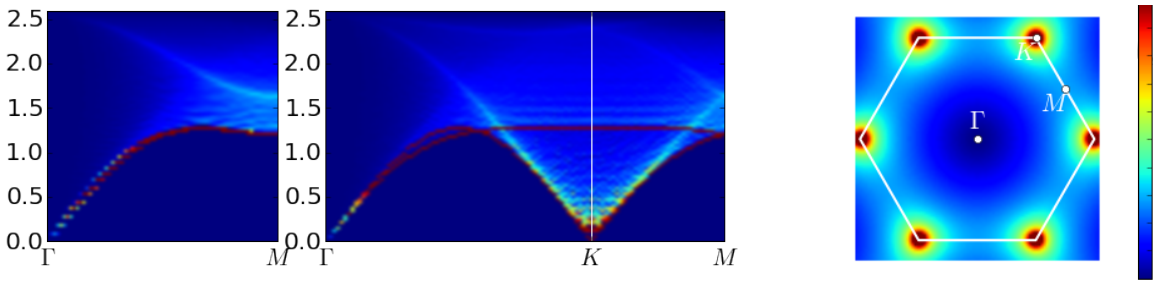
Let $\tilde{M}$ be the diagonal matrix with coefficients ϵ_i and J the matrix $J = \begin{pmatrix} 1_{N_s} & 0 \\ 0 & -1_{N_s} \end{pmatrix}$.

We define $\tilde{v}^\dagger = (\tilde{b}_{\uparrow 1}^\dagger, \ldots, \tilde{b}_{\uparrow N_s}^\dagger, \tilde{b}_{\downarrow 1}, \ldots, \tilde{b}_{\downarrow N_s})$ where the $\tilde{b}_{\sigma j}$ are bosonic operators characterized by a matrix P such that $v = P\tilde{v}$. If P obeys $PJP^\dagger = J$, the bosonic character of the $\tilde{b}$ is preserved. Together with the condition that $P^\dagger M P = \tilde{M}$ is diagonal (this is not a standard matrix diagonalization, since P is not unitary), we get our transformation.

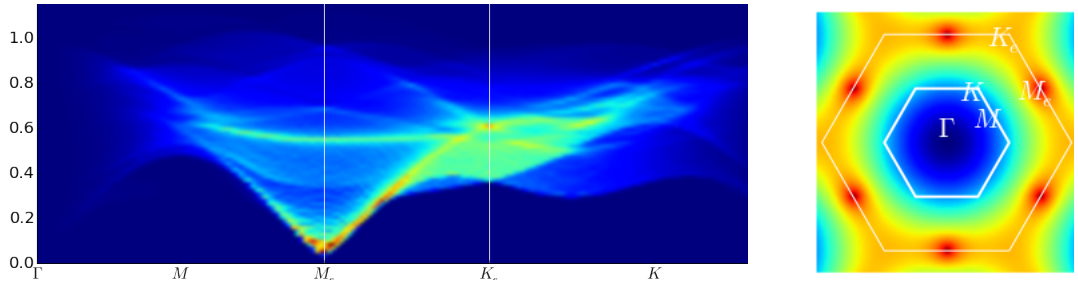
6.3 The possible phases of the ground state

We now have a problem of non interacting bosons. What type of phase can we obtain on a periodic two dimensional lattice ?

- If at least one of the $\epsilon_j < 0$, then the ansatz is unphysical.



(a) Antiferromagnetic triangular lattice with long-range order.



(b) Antiferromagnetic kagome lattice with one of the possible spin liquid ground state (small gap).

Figure 16: Dynamical structure factor $S(\mathbf{q}, \omega)$ (space-time Fourier transform of spin-spin correlation functions $\langle \mathbf{S}(i,t) \cdot \mathbf{S}(i=0,t=0) \rangle$ and the Static (equal time) structure factor $S(\mathbf{q}) = \int_0^\infty \frac{d\omega}{2\pi} S(\mathbf{q}, \omega)$, obtained by Schwinger boson mean-field theory

- If one or several ϵ_i are zero, then we have a Bose condensation. At least one of the $\tilde{b}$ mode is filled (this does not change the energy, but is required to verify the constraint), corresponding to non-zero $\langle b \rangle$ and to a magnetic long range order. Note that the gap only closes in the thermodynamic limit for antiferromagnets (in $1/N$, as for the initial spin model). The dynamical structure factor calculated in such a state is very similar to the results expected from inelastic neutron scattering experiments (see Fig. 16 for the example of the antiferromagnetic triangular lattice), with Bragg peaks and magnon bands.
- If all the ϵ_i are non zero, then we get a unique ground state, even in the thermodynamic limit. It is the vacuum of the $\tilde{b}$ (but not of the b) as seen from Eq. (30). In this case, we say that we describe a $\mathbb{Z}_2$ spin liquid phase of no symmetry is broken by the ansatz. Spin rotational symmetry has been respected by construction. It remains to take care of the lattice symmetries.

The careful reader may be perturbed by the assertion that a $\mathbb{Z}_2$ spin liquid on a torus has a simple ground state degeneracy. Where is the expected 4-fold degeneracy ?

By fixing the ansatz, we have imposed the flux sector (equivalent of the winding sector for the QDM model): the mean-field breaks the local gauge symmetry. However, some gauge symmetry remain, forming the **Invariance Gauge Group** (IGG): the group of gauge

transformations that do not modify the Ansatz. We will come back to it later on. Quantities defined on loops as products of $AA^\dagger AA^\dagger ... A^\dagger$ are gauge invariant. Their phases are called fluxes. The two expected types of quasiparticles are the spinons, effectively present by construction and the visons, which are π $\mathcal{A}$-fluxes on elementary plaquettes. To create a pair of visons on adjacent plaquettes separated by a link ij, we change $\mathcal{A}_{ij} \to -\mathcal{A}_{ij}$. Doing so on a string crossing links (and no site) around the lattice annihilate the visons after moving around the lattice. It does not change any local quantity but changes the flux sector, and the energy of both states are the same in the thermodynamic limit.

There is another way of understanding this construction. We have implicitly supposed periodic boundary conditions for bosons: $b_{i\sigma} = b_{i+L\mathbf{e}_{x/y}\sigma}$. But antiperiodic boundary conditions in a direction means in fact also periodic boundary conditions for spins. However, it changes the flux on any topologically non trivial loop around the lattice in this direction. This is equivalent to changing the sign of $\mathcal{A}_{ij}$ (and $\mathcal{B}_{ij}$) on a loop crossing links in the perpendicular direction.

6.4 The projective symmetry groups

Two ansatze are equivalent if they are related by a gauge transformation. It is equivalent to say that gauge invariant quantities are the same in both ansatze.

Suppose that we have an ansatz such that all gauge invariant quantities are invariant by lattice symmetries $\mathcal{X}$. Applying such a symmetry $X \in \mathcal{X}$ sends it to a gauge equivalent ansatz. Let's call $g_X \in G$ a gauge transformation such that $u_X = g_X X$ does not modify the ansatz (there are as many such gauge transformations as elements in the IGG). The u_X form a projective representation of the symmetry group, called the **projective symmetry group** (PSG) of the ansatz.

Two gauge equivalent ansatze have two different PSG, that are said to be equivalent. All the possible classes of PSG can be determined using only the symmetry group properties and the knowledge of the IGG, called the **algebraic** PSG, and mathematically given by the **second cohomology group** $\mathcal{H}^2(\mathcal{X}, IGG)$. We are going to detail how this can be done. The first such classification was for fermions[47] and then bosons[46]. Before, we go back to the IGG.

The most general IGG is $\mathbb{Z}_2$: changing all b or f operators to their opposite obviously let the A and B operators unchanged. However, in some cases, the IGG is enlarged to $U(1)$. This occurs on bipartite lattices, where only $\mathcal{A}$ are non-zero (only $\mathcal{B}$ is a non interesting case as condensation always occurs). Then, changing the phase by θ on a sublattice and $-\theta$ on the other does the work. We focus here on a $\mathbb{Z}_2$ IGG.

We take the example of a triangular lattice, with symmetries generated by translations $\mathcal{T}_1$ and $\mathcal{T}_2$, rotations $\mathcal{R}_6$ and axial symmetry σ (see Fig. 17).

$$\mathcal{T}_1 : (r_1, r_2) \to (r_1 + 1, r_2) \tag{31a}$$
$$\mathcal{T}_2 : (r_1, r_2) \to (r_1, r_2 + 1) \tag{31b}$$
$$\mathcal{R}_6 : (r_1, r_2) \to (r_1 - r_2, r_1) \tag{31c}$$
$$\sigma : (r_1, r_2) \to (r_2, r_1). \tag{31d}$$

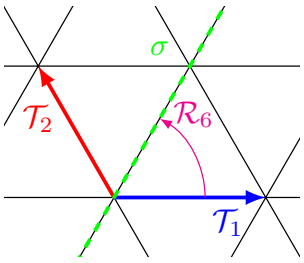


Figure 17: Symmetries on a triangular lattice.

The relations that allow to write any product of transformation in an ordered way are:

$$\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1 \tag{32a}$$
$$\sigma^2 = I \tag{32b}$$
$$\mathcal{R}_6^6 = I \tag{32c}$$
$$\mathcal{T}_1\mathcal{R}_6 = \mathcal{R}_6\mathcal{T}_2^{-1} \tag{32d}$$
$$\mathcal{T}_2\mathcal{R}_6 = \mathcal{R}_6\mathcal{T}_1\mathcal{T}_2 \tag{32e}$$
$$\mathcal{T}_1\sigma = \sigma\mathcal{T}_2 \tag{32f}$$
$$\mathcal{R}_6\sigma\mathcal{R}_6 = \sigma. \tag{32g}$$

Each of these relations will give us a constraint on the transformations in the algebraic PSG, as they each give a different way to apply the identity, i.e. an element of the IGG.

We begin by the first constraint, coming from $\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1$. As $\mathcal{T}_1\mathcal{T}_2\mathcal{T}_1^{-1}\mathcal{T}_2^{-} = I$, the successive application of PSG elements associated to the four translations on the left must send the ansatz to itself: it must belong to the IGG.

$$g_1\mathcal{T}_1 g_2\mathcal{T}_2\mathcal{T}_1^{-1}g_1^{-1}\mathcal{T}_2^{-1}g_2^{-1} \in IGG. \tag{33}$$

In terms of effects on the phase of partons, it becomes:

$$\phi_1(r) + \phi_2(\mathcal{T}_1^{-1}r) - \phi_1(\mathcal{T}_2^{-1}r) - \phi_2(r) = p_1\pi, \tag{34}$$

with $p_1 = 0$ or π and $\phi_X(r)$ is the phase of the gauge transformation g_X on site r.

We can further simplify this result as we only want a specific PSG in each equivalence class. Thus, we can apply a gauge transformation G to the ansatz chosen especially to cancel some phases in the last equation. If $g_X X$ was in the PSG of the initial ansatz, $Gg_X XG^{-1}$ is in the new PSG. In other words, the effect of a gauge transformation G on the PSG is:

$$\phi_X(r) \to \phi_G(r) + \phi_X(r) - \phi_G(X^{-1}(r)). \tag{35}$$

Step by step (by increasing x), we can determine a gauge transformation that cancels all $\phi_1(r)$:

$$\phi_G(x, y) = -\sum_{i=0}^{x} \phi_1(i, y). \tag{36}$$

With periodic boundary conditions, we have $\sum_{i=0}^{L-1} \phi_1(i,y) = 0$ and no correction is needed when crossing the link $L-1, 0$. For antiperiodic boundary conditions, $\sum_{i=0}^{L-1} \phi_1(i,y) = \pi$, and $\phi_1(0,y)$ cannot be cancelled but has to be π.

We fix periodic boundary consitions and fix $\phi_1(r) = 0$. Eq. (34) is now

$$\phi_2(x-1,y) - \phi_2(x,y) = p_1\pi, \tag{37}$$

that is solved as

$$\phi_2(x,y) = \phi_2(0,y) + p_1\pi x \tag{38}$$

To simplify ϕ_2, we define a new gauge transformation:

$$\phi_G(x,y) = \sum_{i=0}^{x} \phi_1(i,y) + \sum_{j=0}^{y} \phi_2(0,j). \tag{39}$$

that has the same effect as previously on ϕ_1 but now, remove the first term in Eq. (38).

Finally, the $\mathbb{Z}_2$-PSG containing $\mathcal{T}_1$ and $\mathcal{T}_2$ on the triangular lattice (also true on the square lattice) divide into two equivalence classes depending on p_1:

$$\phi_1(x,y) = 0, \qquad \phi_2(x,y) = p_1\,\pi\,y. \tag{40}$$

The same calculations can be performed for all equations in (32). For completness, I give the result for $\phi_{\mathcal{R}_6}$ where each of the previous class is still subdivided into several classes depending on an integer k:

$$\phi_1(x,y) = 0, \qquad \phi_2(x,y) = p_1\,\pi\,y, \qquad \phi_{\mathcal{R}_6}(x,y) = p_1\pi\frac{2xy + y(y+1)}{2} + k\frac{\pi}{6}. \tag{41}$$

7 Conclusion

We have seen the different ways to classify phases depending on their entanglement properties, and to further classify them according to the symmetries they break or respect. $\mathbb{Z}_2$ spin liquids have been obtained in several simple toy models without any symmetry; the Rokhsar-Kivelson dimer model and the toric code. They allow to get simple pictures of what is an anyon, what is entanglement entropy. Considering realistic spin model remains a challenge, partly resolved by the projective symmetry group approach in parton constructions.

Bosonic and fermionic PSGs are related, as a fermionic spinon can be seen as a composite particle formed by a bosonic spinon and a vison[26]. However, PSG of parton construction do not fully classify the $\mathbf{Z}_2$ spin liquids. The complete classification requires to consider symmetry fractionalization as detailed in [12]. Actually, the PSGs of last section only treat the symmetry fractionalization of one type of anyons: the spinons. To be complete, a group cohomology class must be assigned to each anyon, with constraints from the fusion rules.

- In Chern Simons U(1) spin liquids (simu by Donna Sheng), $\gamma = \ln 2/2$. Why ???? $ln2$ should be the minimal ?

- simple intro to cohomology groups ?
- Can a toric code state with a e particle be a ground state ? No (not possible with PBC). With 2 e ? Yes (we favor two positive A_s for example). This should statistics for the QP that depend on the path... It breaks the local constraints. Answer: the state without any QP are representatives of the same classe of states (LU trnasformation can link them) and are the only one described by Chen et al.

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