Topology and Homotopy

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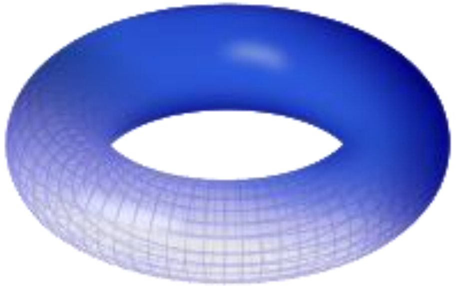
Introduction: Homeomorphism

Topology aim at classify objects in classes where objects can be transformed one to another using a continuous deformation



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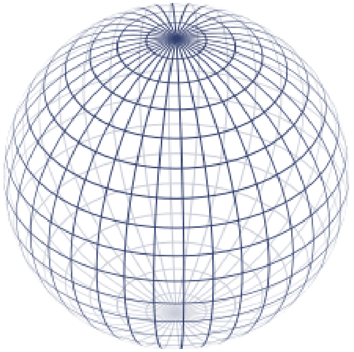

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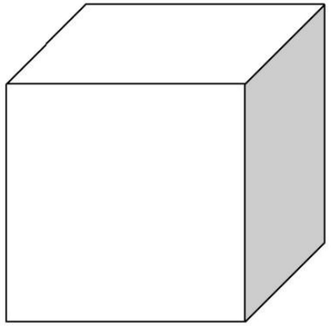
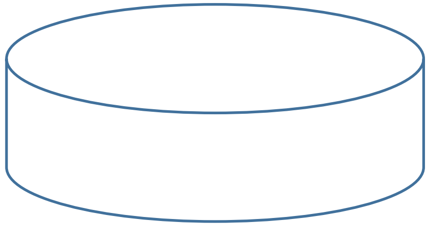




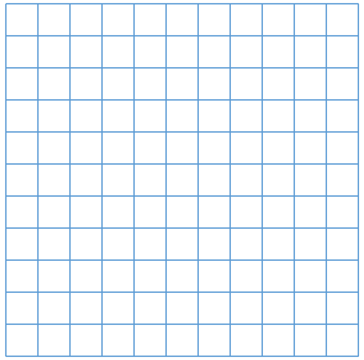
$\not\approx$

Definition: **Homeomorphism** $\approx$

Two objects are homeomorph ($S_1 \approx S_2$) if there is a continuous transformation from one to another

Introduction: Homeomorphism in solid state physics
Example of dislocations

Perfect crystal



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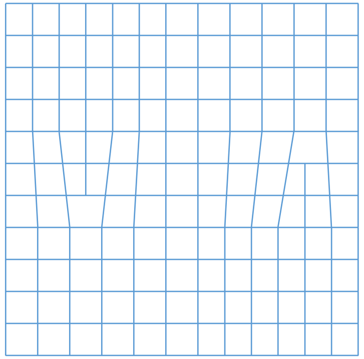
With 2 opposite dislocations
Topology is preserved



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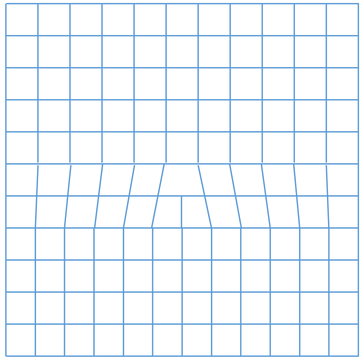
D1 : Crystal with
a dislocation



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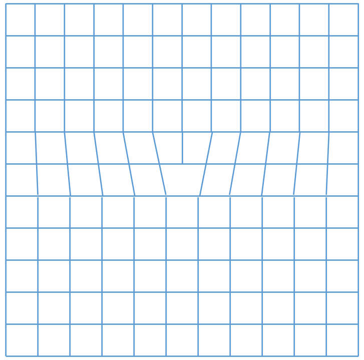
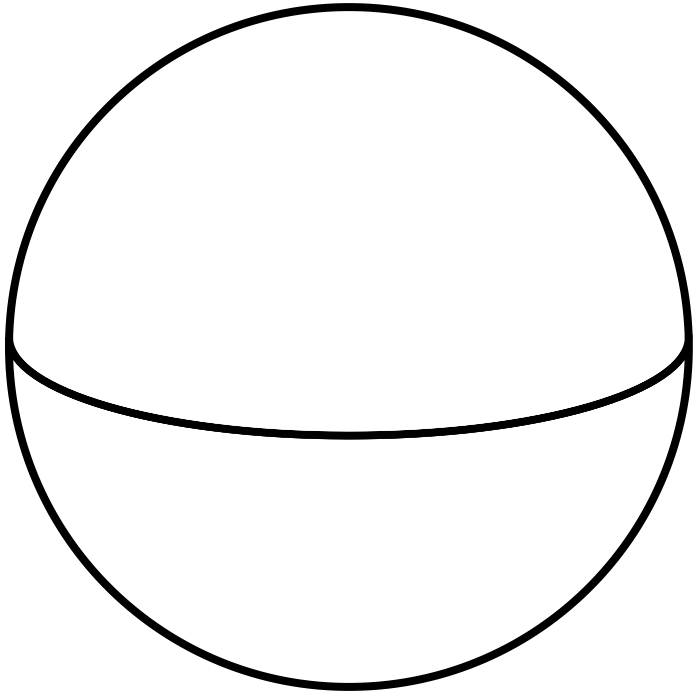
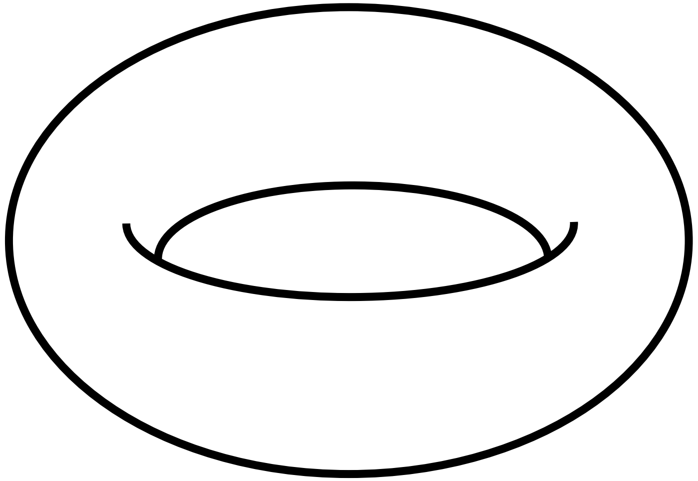
D2 : Crystal with
another dislocation



Introduction: Objectives of the algebraic topology





How to define formaly the topological difference?
How to define formally a hole?
Can we define topologically invarient quantitites?

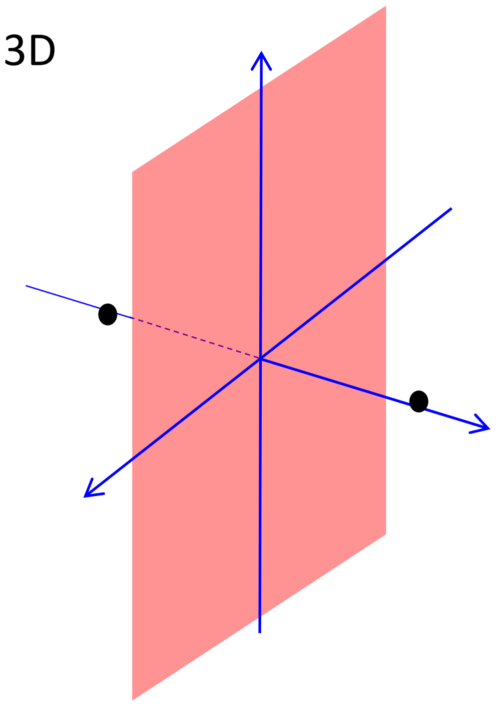
Outline

1. **Introduction:** homeomorphism
2. **How to catch a topological defect or texture**
3. **Homotopy and homotopy group**
4. **Geometrical space and order parameter space**
5. **Topological defects and topologically stable configurations**

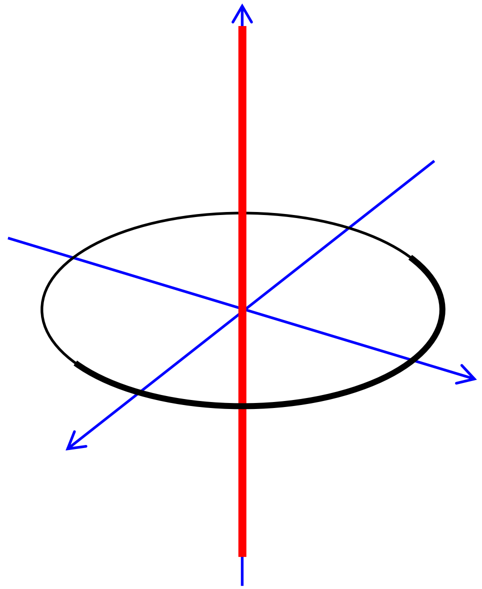
How to catch a topological defect

The prey and the hunter rule
A contour surronds the defect to catch it.

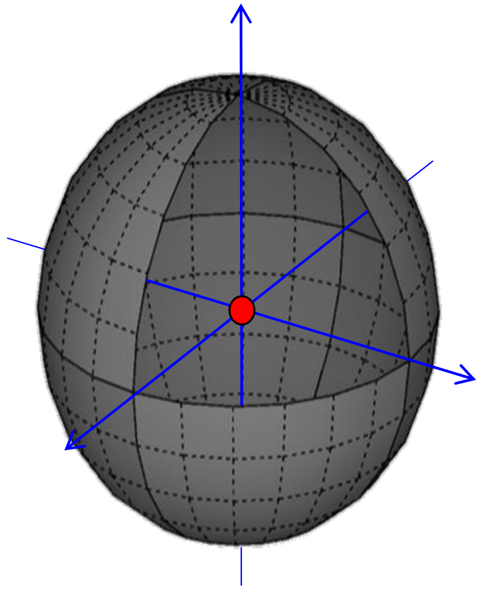
Example in 3D



Surfacic defect



Linear defect



Punctual defect

Hunter's rule

$$d' + r = d - 1$$

d': Defect dimension

d : Space dimension

r: Contour dimension

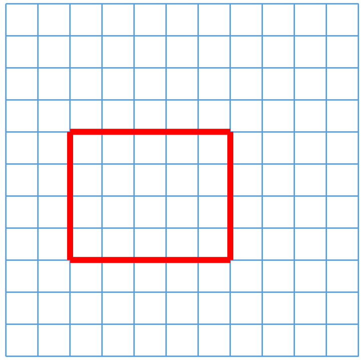
Applications of the hunter's rule to dislocations

$$d' + r = d - 1$$

$$d' = 0$$
$$d = 2$$
$$r = 1$$

Perfect crystal

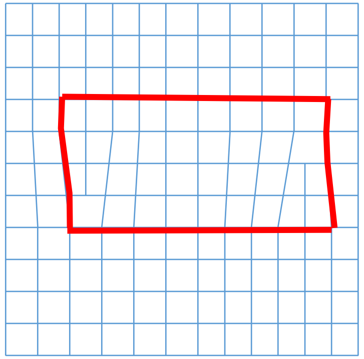
$\vec{b} = \vec{0}$



Any contour $(n \times p \times n \times p)$ is closed

$\approx$

With 2 opposite dislocations
Topology is preserved

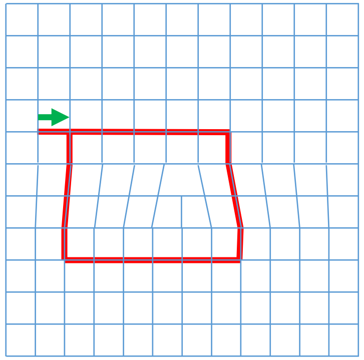


$\vec{b} = \vec{0} = \vec{b}_{D1} + \vec{b}_{D2}$

$\not\approx$

D1 : Crystal with a dislocation

$\vec{b}_{D1} = a\vec{x}$



The same contour is not closed

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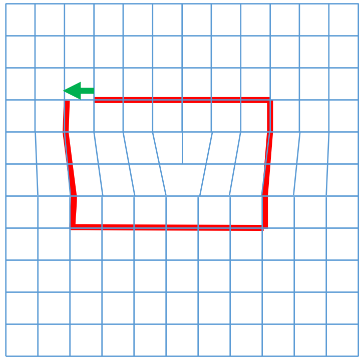
$\not\approx$

D2 : Crystal with another dislocation

$\vec{b}_{D2} = -a\vec{x}$



The result is different again

How can we characterize the dislocation?
-> Burgers vector
Topological invarient

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Homotopies

1. Properties of contours Explanation for 1D contours γ dans un espace $\mathcal{E}$

Definition: Path $\gamma_{x_0 x_1}$

$\gamma_{x_0 x_1}: [0,1] \mapsto \mathcal{E}$ is a continuous application of $[0,1]$ over the space $\mathcal{E}$ so that $\gamma(0) = x_0$ and $\gamma(1) = x_1$

Definition: Closed path or contour γ in x_0

$\gamma: [0,1] \mapsto \mathcal{E}$ is a continuous application of $[0,1]$ over the space $\mathcal{E}$ so that $\gamma(0) = \gamma(1) = x_0$

Definition: Homotopy ~

γ_1 and γ_2 are homotope ($\gamma_1 \sim \gamma_2$) if is there is a continous transformation from one to another.

Definition: Class of contour $[y]$

$\forall \gamma_1$ and $\gamma_2 \in [\gamma], \gamma_1 \sim \gamma_2$

Definition: Composition .

The composition $\gamma_1 . \gamma_2$ of two contours is a contour. The composition commutes: $\gamma_1 . \gamma_2 \sim \gamma_2 . \gamma_1$

Additionnal property : $[\gamma_1 . \gamma_2] = [\gamma_1] . [\gamma_2]$

Independence on the starting point

In a continuous space, there is always a path $\gamma_{x_0 x_1}$ that links to points. A contour γ starting from x_0 is homotope to a contour $\gamma' = \gamma_{x_1 x_0} . \gamma . \gamma_{x_0 x_1}$ starting from x_1

Neutral contour A neutral contour c is such that $\forall \gamma, \gamma . c \sim \gamma$.

If $\gamma \sim c$, then γ is neutral.

Reverse contour $\forall \gamma, \exists \gamma^{-1}$ so that $\gamma . \gamma^{-1} = c$.

If $\gamma \sim \gamma^{-1}$ then $\gamma \sim c$

Homotopy group

Definition: Homotopy group

The homotopy group is the group of the classe of paths

$$\pi_1(\mathcal{E}) = \{\ [\gamma]\ \}$$

Dimension of the contour → 1

Space or volume → $\mathcal{E}$

Ensemble of class of contour in $\mathcal{E}$ → $\{\ [\gamma]\ \}$

Special case: If $\forall\gamma$ in $\mathcal{E}, \gamma \sim c$ (or $\gamma \in [c]$),

the homotopy group $\pi_1(\varepsilon) = \{\ [c]\ \}$ contains a single element.

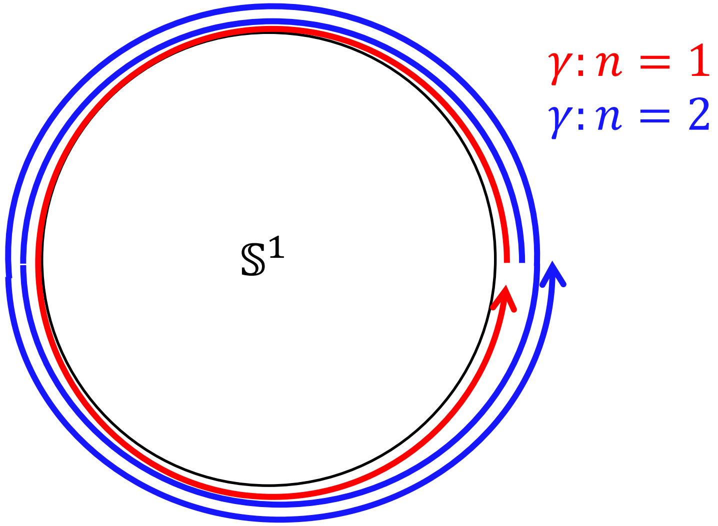
It is said to be trivial.

Ex: 2D plane $\mathbb{P} = \mathbb{R}^2$: $\pi_1(\mathbb{P})$ is trivial

Homotopy group of the circle $\mathcal{E} = \mathbb{C} = \mathbb{S}^1$

Degree $\deg(\gamma)$ of a contour: number of tour around the circle.

$$\deg(\gamma) \in \mathbb{Z}$$



Properties

- $\deg(c) = 0$
- $\deg(\gamma) = -\deg(\gamma^{-1})$
- $\deg(\gamma_1 . \gamma_2) = \deg(\gamma_1) + \deg(\gamma_2)$

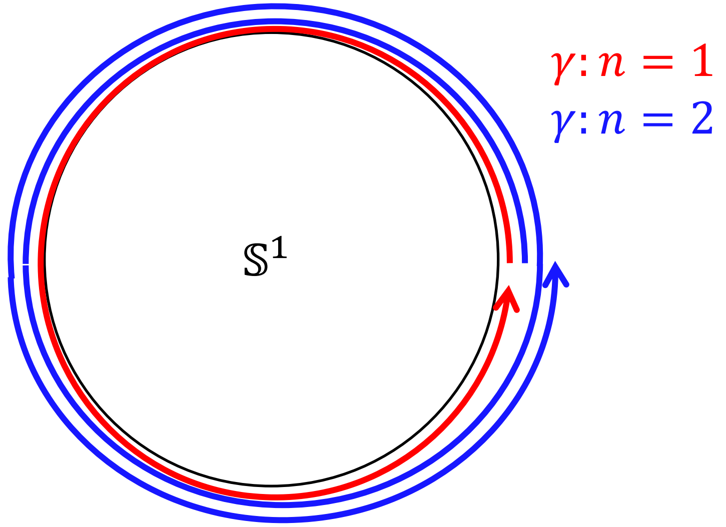
The degree is called **topological invarient**

$$n = \deg(\gamma)$$

Homotopy group of the circle $\mathcal{E} = \mathbb{C} = \mathbb{S}^1$

Example: $\mathbb{S}^1 = \mathbb{C}_{|z|=1}$

$z \in \mathbb{S}^1 \to z = e^{i\theta}$



Any contour γ is homotope to $\gamma_n(t) = e^{2i\pi nt}$
with $t \in [0,1]$ et $n \in \mathbb{Z}$

$$\deg(\gamma) = n = \frac{1}{2\pi}\oint_\gamma dz = \int_0^1 n dt = n$$

Homotopy group of the circle $\mathcal{E} = \mathbb{C} = \mathbb{S}^1$

Any contour is defined by its degree or topological invarient

We can build **the classes of contour** $[\boldsymbol{\gamma_n}]$ such that all the contours of the class share the same degree n.

$$\pi_1(\mathbb{S}^1) = \{\, [\gamma_n], n \in \mathbb{Z}\}$$

Since $\mathbb{Z}$ fully defines the homotopy group (isomorphism between $\{\, [\gamma_n], n \in \mathbb{Z}\}$ and $\mathbb{Z}$) the homotophy group is usually written using the group of topological invarients:

$$\pi_1(\mathbb{S}^1) = \mathbb{Z}$$

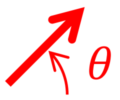
The composition operation between contours becomes an addition in $\mathbb{Z}$

$$\deg(\gamma_1.\gamma_2) = \deg(\gamma_1) + \deg(\gamma_2)$$
$$n(\gamma_1.\gamma_2) = n(\gamma_1) + n(\gamma_2)$$

Homotopy group of the circle $\mathcal{E} = \mathbb{C} = \mathbb{S}^1$

Vortex and antivortex

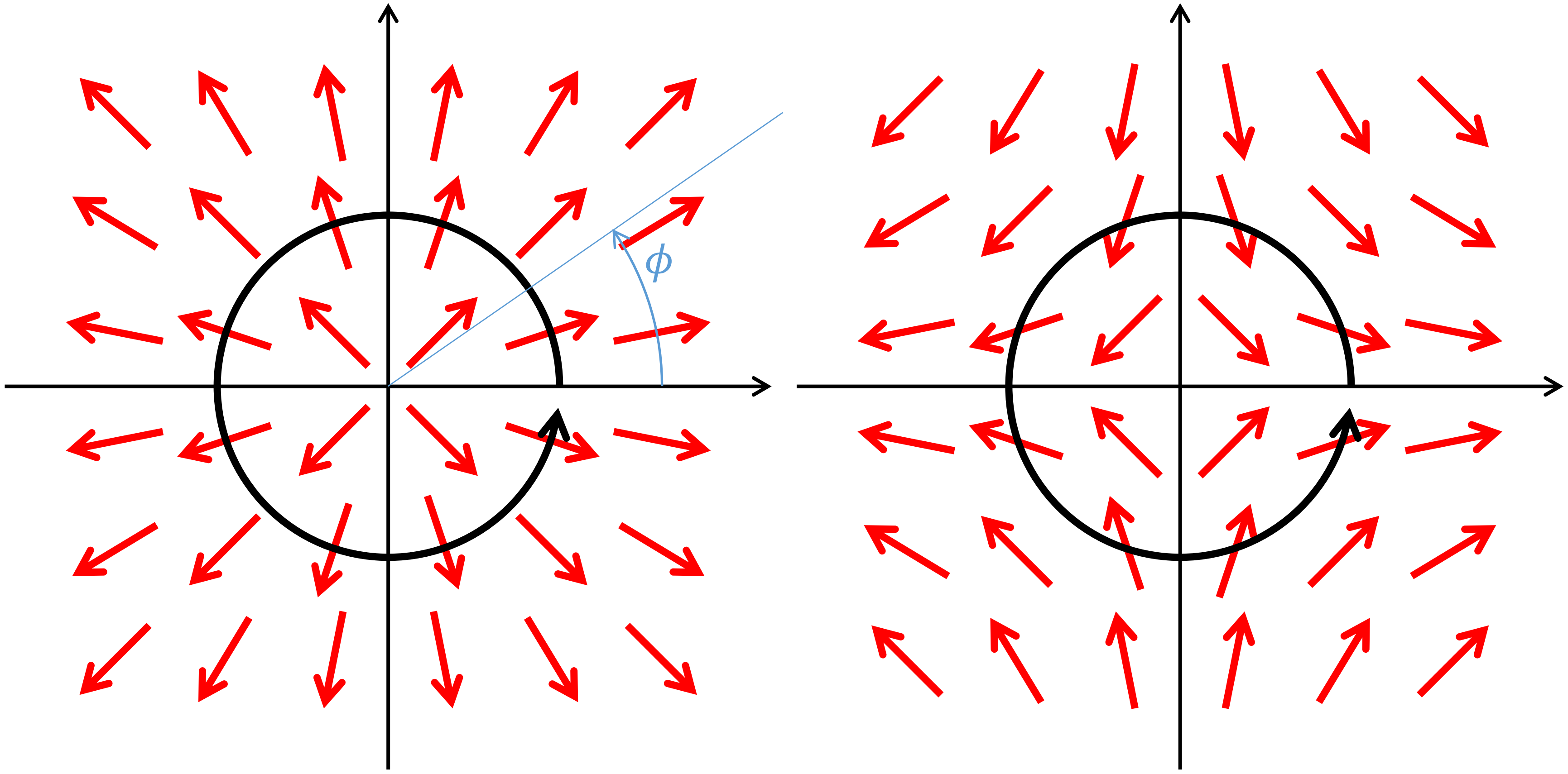
We consider XY spins that live on $\mathbb{S}^1$:
Spins are defined by an angle θ.



Hunter's rule:
$d' + r = d - 1$

The defect is puntual $d' = 0$
Space is 2D $d = 2$

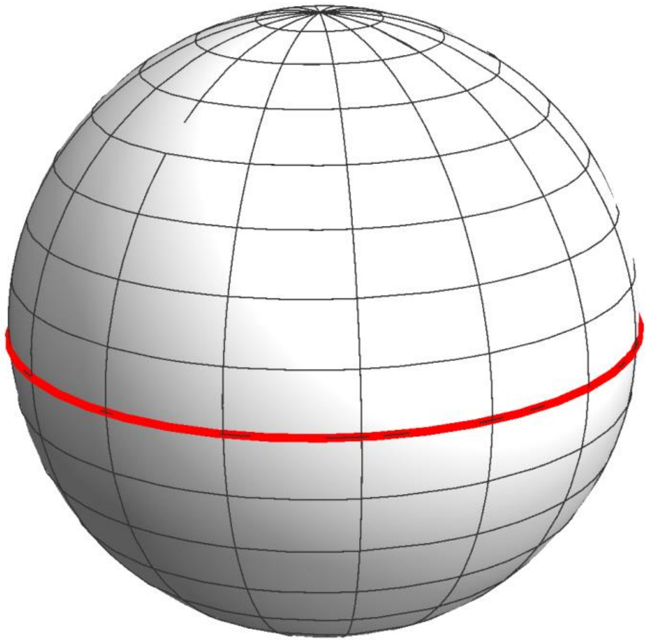
Contour is $r = 1$ (circle)
Described by an angle ϕ $(0 \to 2\pi)$



Vortex: $\theta = \phi$
$n = 1$

AntiVortex: $\theta = -\phi$
$n = -1$

Homotopy group of the sphere $\mathbb{S}^2$
1. First group $\pi_1(\mathbb{S}^2)$



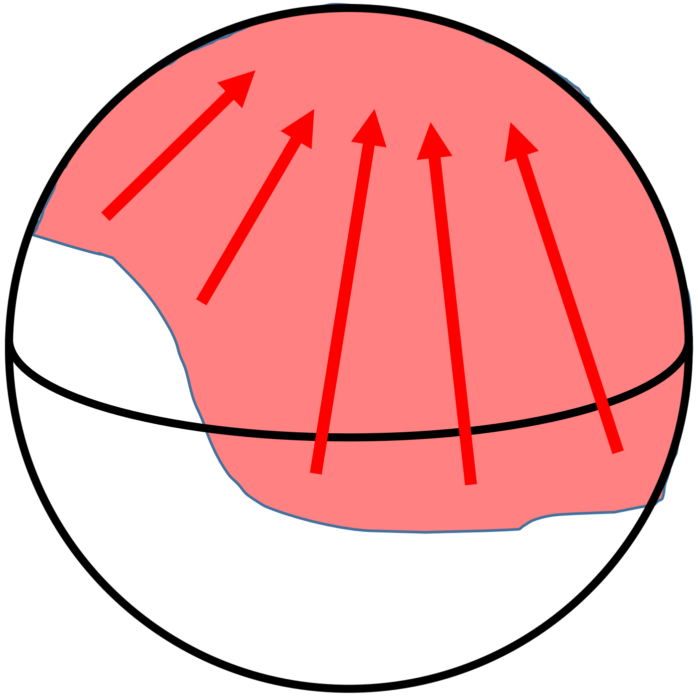
Any contour is homotope to a single point (neutral contour):

The homotopy group is trivial

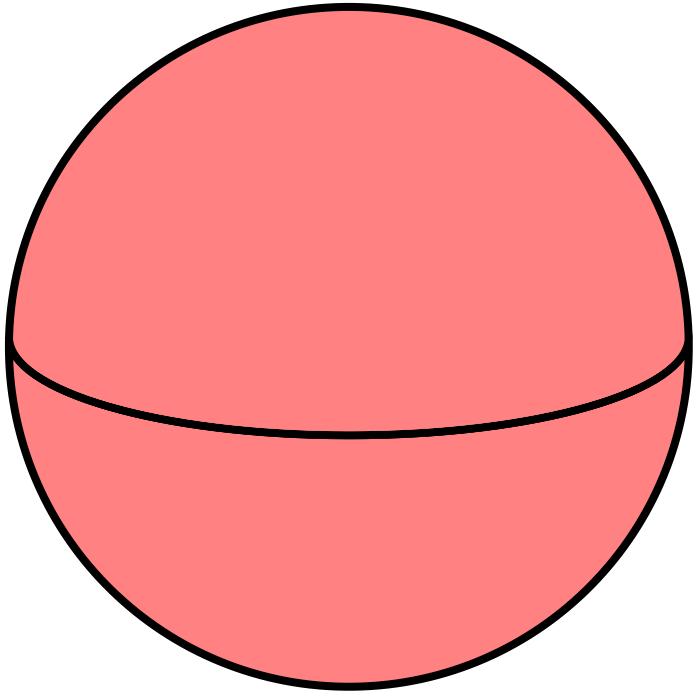
$$\pi_1(\mathbb{S}^2) = \{c\} = 0$$

Homotopy group of the sphere $\mathbb{S}^2$
2. Second group $\pi_2(\mathbb{S}^2)$

A contour corresponds to a compact surface



This contour is homotope to a point
Trivial or neutral element.



First non trivial element: the sphere is fully covered.

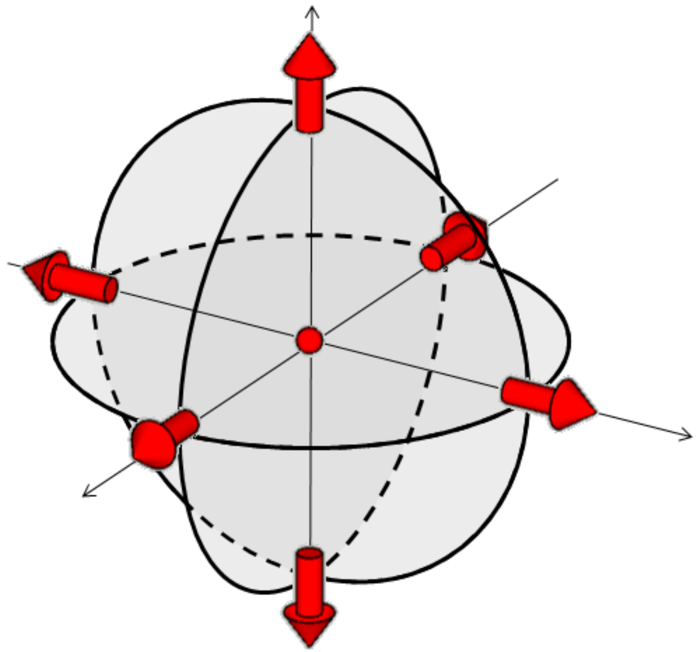
$$\pi_2(\mathbb{S}^2) = \mathbb{Z}$$

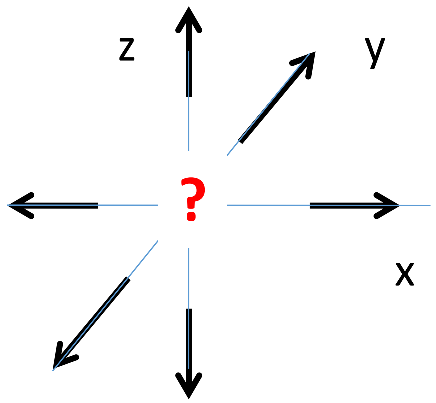
Topological invarient: nb of time the sphere is wrapped.

Homotopy group of the sphere $\mathbb{S}^2$
2. Second group $\pi_2(\mathbb{S}^2)$

Example of a Bloch point (magnetic monopole)

We consider Heisenberg spin: they live on $\mathbb{S}^2$ (two angles θ and ϕ)





Hunter's rule:

$d' + r = d - 1$

The defect is puntual $d' = 0$
Space is 3D $d = 3$

Contour is $r = 2$ (sphere)

The spin texture found on the spherical contour corresponds to a portion of the $\mathbb{S}^2$.

Around the Bloch point, all the spin orientations are found. The spin $\mathbb{S}^2$ sphere is fully covered.

$n = \pm 1$

First description: Feldtkeller Z. Angew. Phys. (1965)

Homotopy group of the spheres $\mathbb{S}^p$

General results:

$$\pi_n(\mathbb{S}^p) = 0 \text{ if } n < p$$

$$\pi_n(\mathbb{S}^p) = \mathbb{Z} \text{ if } n = p$$

topological invarient: $\frac{1}{|\mathbb{S}^p|} \int_{\gamma_n} ds$

(number of time the sphere is covered)

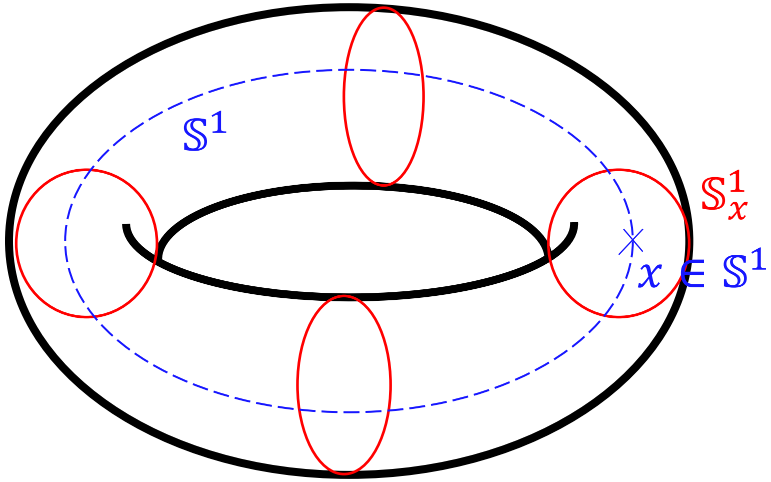
	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z{\times}Z_{12}$	Z_2^2	Z_2^2	$Z_{24}{\times}Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}{\times}Z_{12}{\times}Z_2$	$Z_{84}{\times}Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}{\times}Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}{\times}Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z{\times}Z_{120}$

A. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, 2002 and online available at http://www.math.cornell.edu/~hatcher).

Graph source: Wikipedia

Homotopy group of the torus $\mathbb{T}^2$

The torus corresponds to the multiplication of two circles

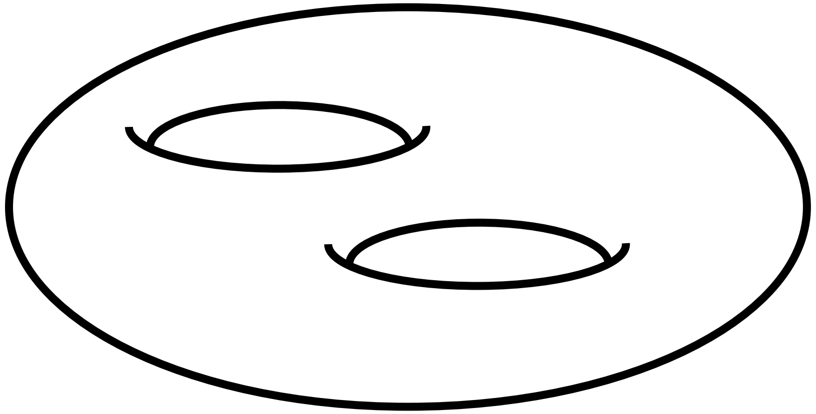


$$\mathbb{T} = \bigcup_{x \in \mathbb{S}^1} \mathbb{S}_x^1 \approx \mathbb{S}^1 \times \mathbb{S}^1$$

$$\pi_1(\mathbb{T}) = \mathbb{Z} \times \mathbb{Z}$$

Topological invarient: (n_1, n_2)

Bonus question: Homotopy group of a sphere with many holes



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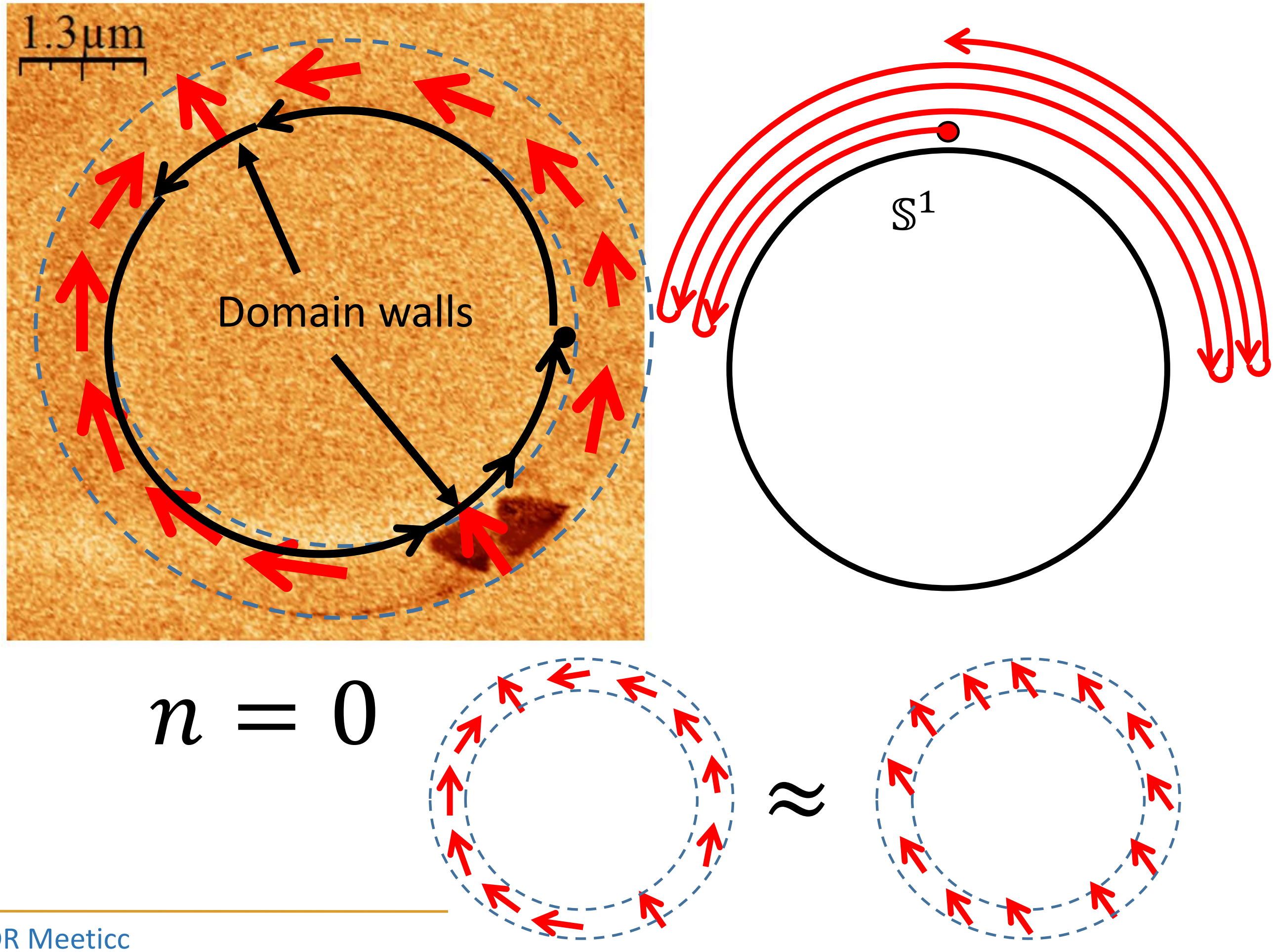
Projection of the geometrical space over the order parameter space

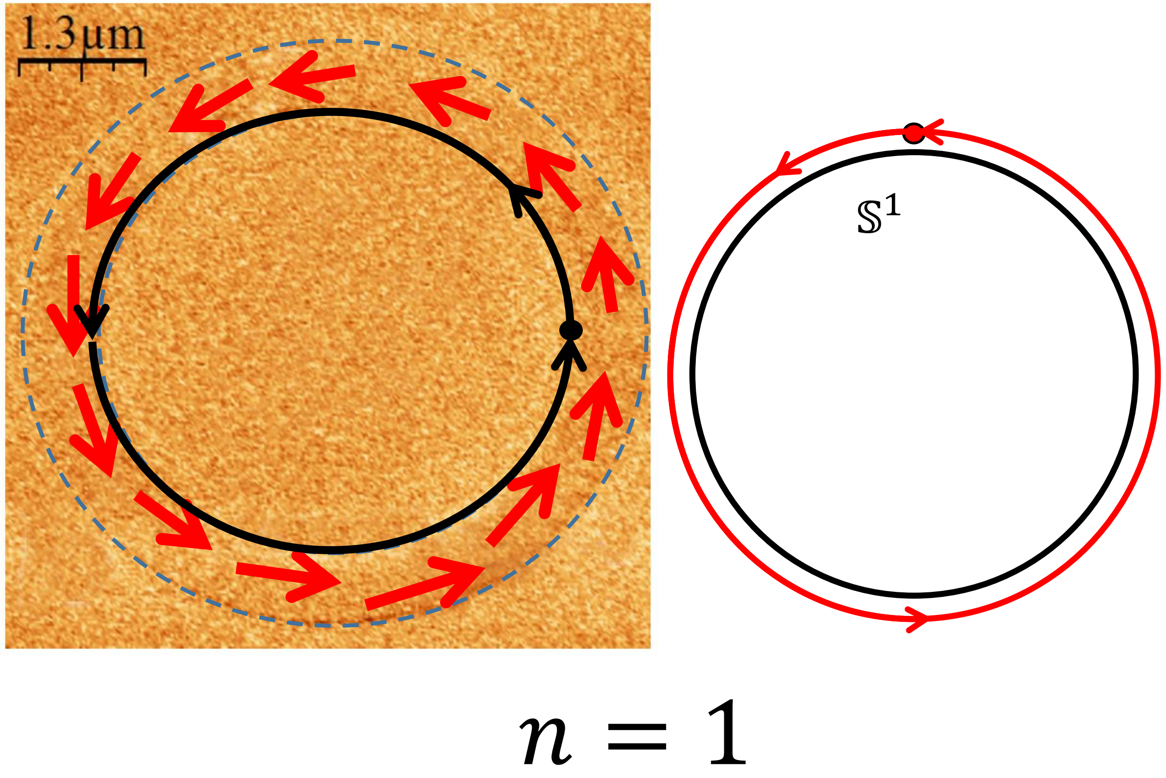
1. Easy situation: the sample has the same geometry as the order parameter space.

Ex: soft magnetic ring

The dipolar interaction forces the magnetization along the ring ($\vec{m}.\vec{n} = 0$).
Magnetization is $\mathbb{S}^1$-like

Magnetic force microscopy images of NiFe magnetic ring
J.Y. Chauleau (LPS, 2011)

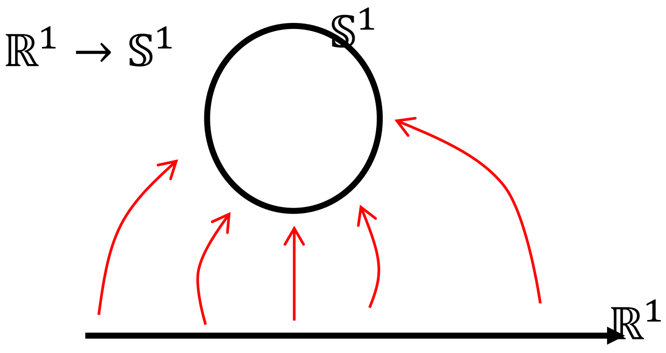


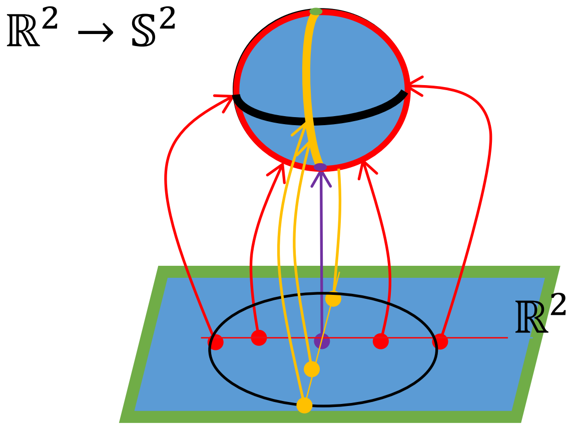


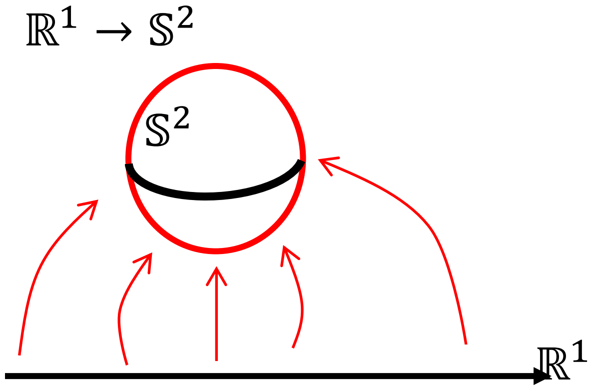
Projection of the geometrical space over the order parameter space

2. The geometrical and order parameter spaces are different

Isomorphism is possible if boundary conditions are uniform







$\mathbb{R}^3 \to \mathbb{S}^2$???

$\Rightarrow$ **Follow the path and look to its trace on the order parameter space.**

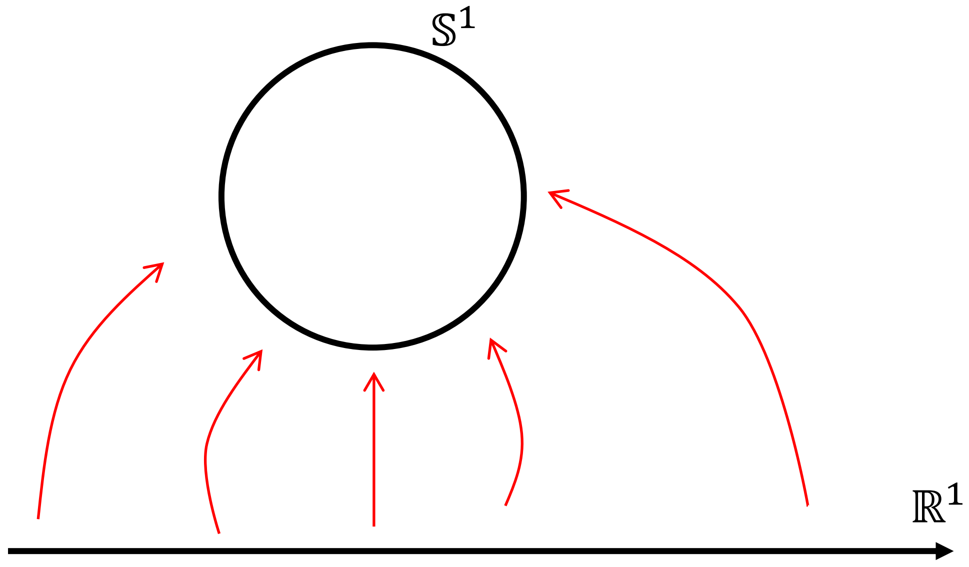
$\Rightarrow$ The trace is a contour of the same dimension as the geometrical space

Look for $\pi_d(\mathbb{S}^p)$ homotopy group

- d: geometrical space dimension
- p: order parameter space dimension

$\mathbb{R}^1 \to \mathbb{S}^1$

We consider a 1D space $\mathbb{R}^1$ with some spins $\mathbb{S}^1$
$\Rightarrow$ We look to the trace of the order parameter on $\mathbb{S}^1$ when we move along $\mathbb{R}^1$.



It corresponds to the developement performed to fine the topological invarient of $\mathbb{S}^1$.

The homotopy group is analogue to the homotopy group $\pi_1(\mathbb{S}^1) = \mathbb{Z}$

Example: 360° domain wall

The topological invarient is the (signed) number of 360° turns

If $\vec{S}(x) = (\cos\theta(x), \sin\theta(x))$

$$n = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{d\theta}{dx}(x)dx = \frac{1}{2\pi}\int_{\theta(-\infty)}^{\theta(+\infty)} d\theta$$

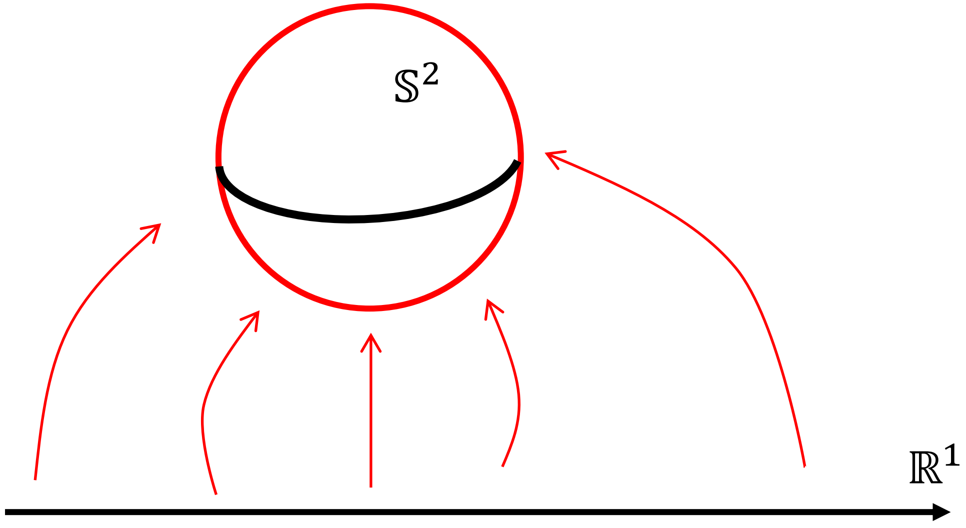
A 360° domain wall in $\mathbb{S}^1$ cannot collapse
Collapse require new degree of freedom
(change the norm of the spin or $\mathbb{S}^2$ spins)

$\mathbb{R}^1 \to \mathbb{S}^2$

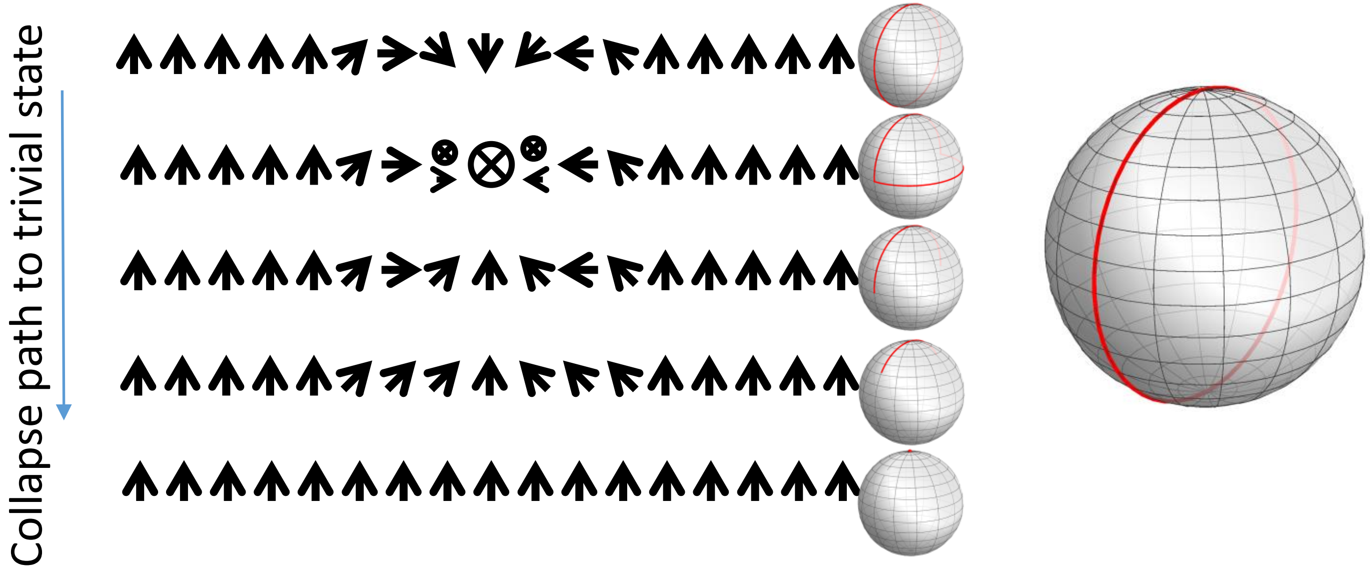
Same work using Heisenberg spins $\mathbb{S}^2$
$\Rightarrow$ We look to the trace of the order parameter on $\mathbb{S}^2$ when we move along $\mathbb{R}^1$.

The order parameter space is partially covered

The homotopy group is analogue to the homotopy group $\pi_1(\mathbb{S}^2) = 0$



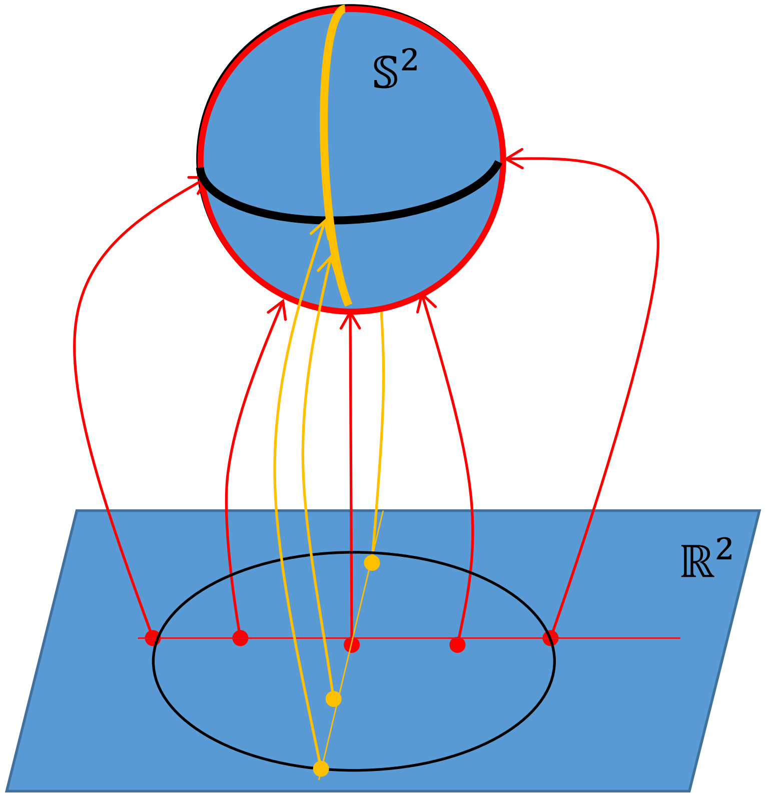
Example: 360° domain wall



$\Rightarrow$ Adding a new degree of freedom removes the topological protection

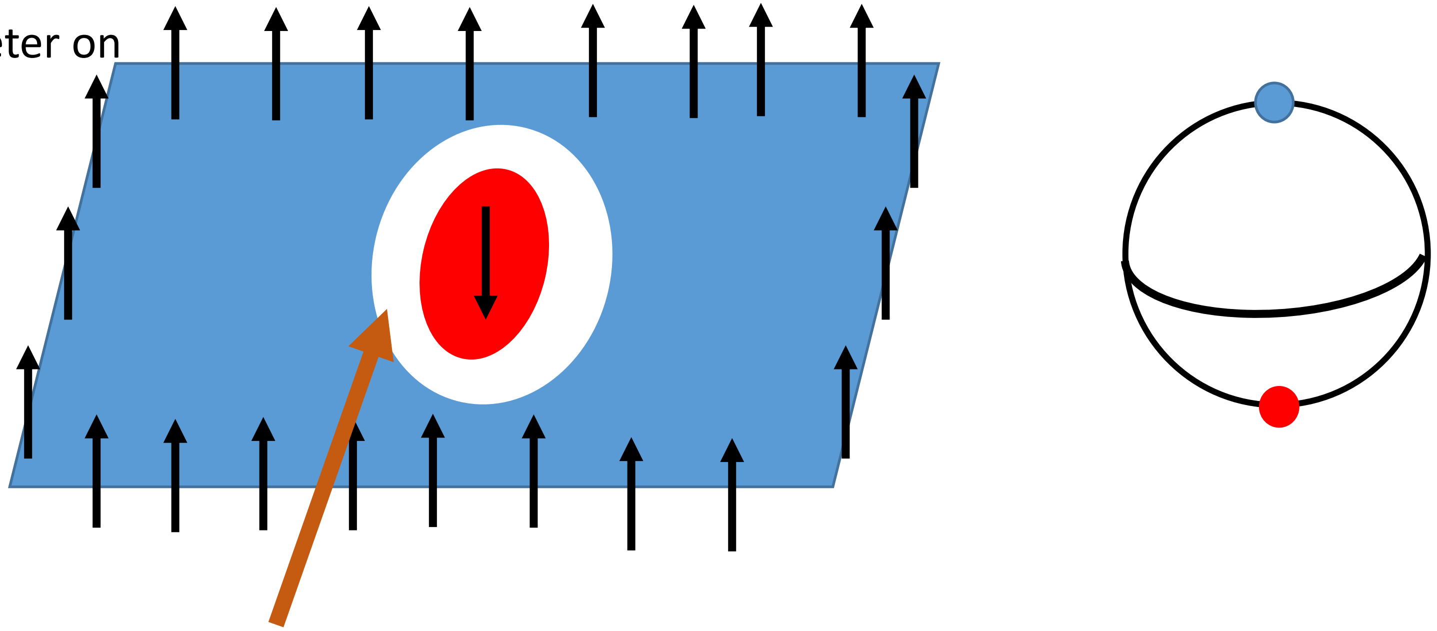
$\mathbb{R}^2 \to \mathbb{S}^2$

2D space using Heisenberg spins $\mathbb{S}^2$
$\Rightarrow$ We look to the trace of the order parameter on $\mathbb{S}^2$ when we move along $\mathbb{R}^2$.

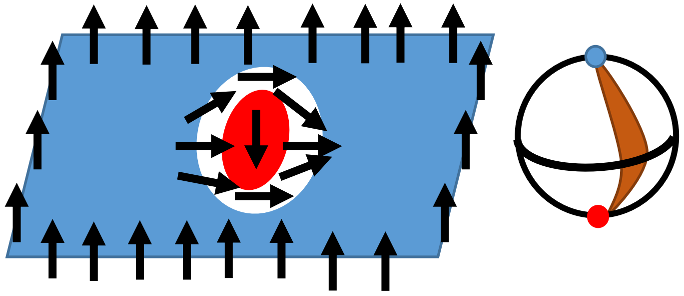


The homotopy group is analogue to the homotopy group $\pi_2(\mathbb{S}^2) = \mathbb{Z}$

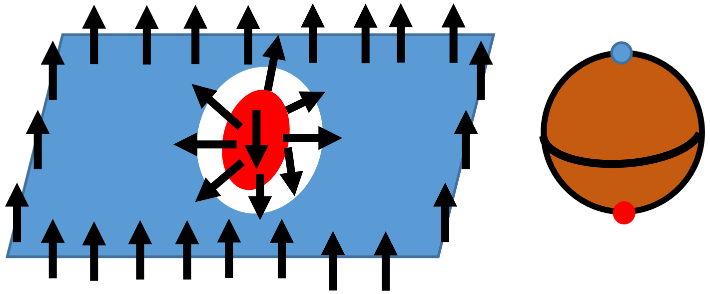
Example: Magnetic bubbles and skyrmions



The key is in the transition zone

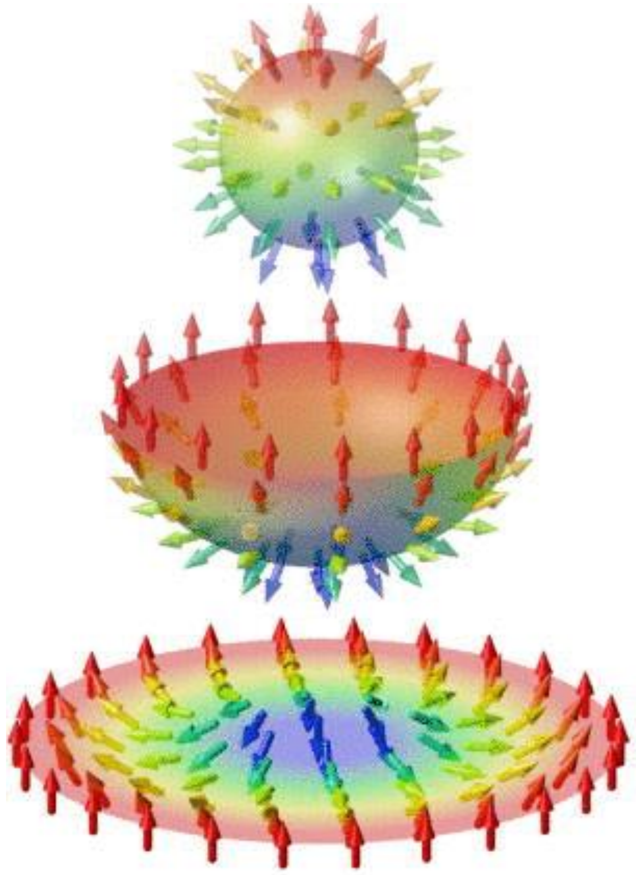


Non topological solution
$n = 0$



Topological solution
$n = 1$

$\mathbb{R}^2 \rightarrow \mathbb{S}^2$: Magnetic skyrmions



The topological invarient is the number of time the unit sphere is covered

$$n = \frac{1}{4\pi} \iint \vec{m}.\left(\frac{\partial \vec{m}}{\partial x} \times \frac{\partial \vec{m}}{\partial y}\right) d^2r$$

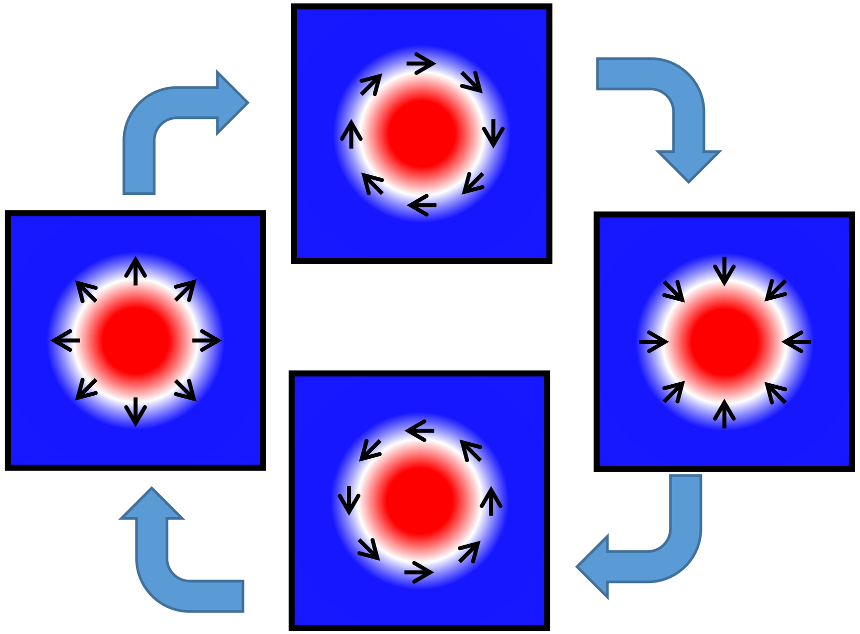
For a centrosymmetric texture:

$$n = \frac{1}{4\pi} \int \sin\theta \frac{d\theta}{dr} dr \int \frac{d\phi}{d\varphi} d\varphi = pW$$

$2p = m_z(r=0) - m_z(r=\infty)$
$p \in \{-1,0,1\}$

Winding number W
$W \in \mathbb{Z}$
(topological invariant in $\pi_1(\mathbb{S}^1)$)

- Topology does not depend on the chirality
- Néel and Bloch skyrmions belong to the same homotopy class

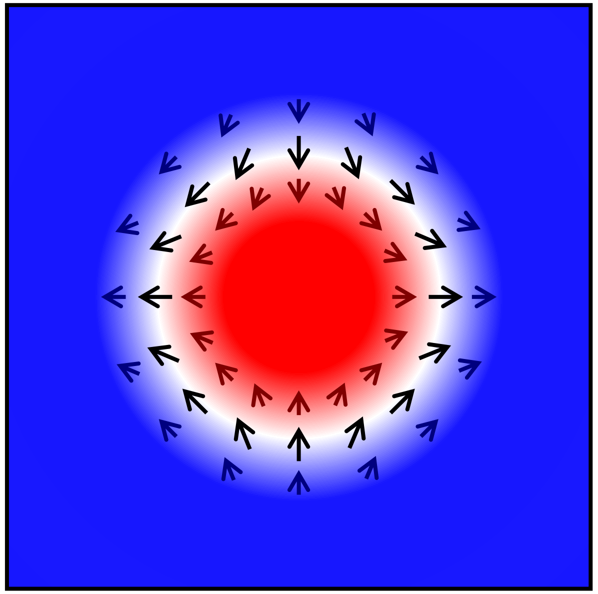


$\mathbb{R}^2 \rightarrow \mathbb{S}^2$: Magnetic skyrmions and topological elements

Antiskyrmion

$p = 1; W = -1$

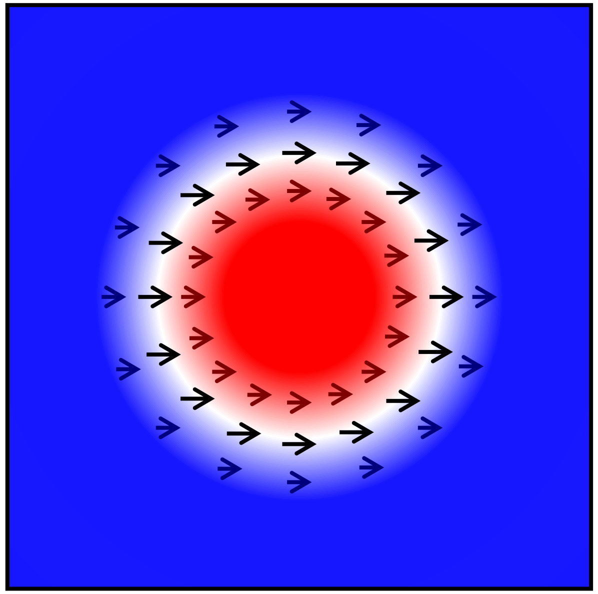
$n = -1$



Trivial element:

$p = 1; W = 0$

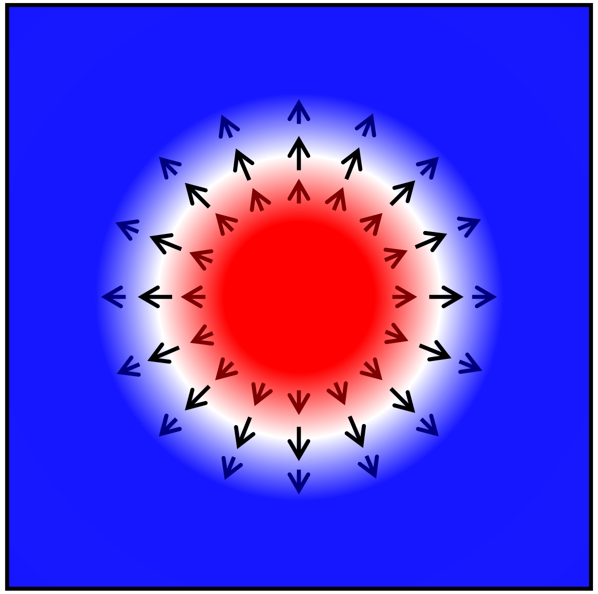
$n = 0$

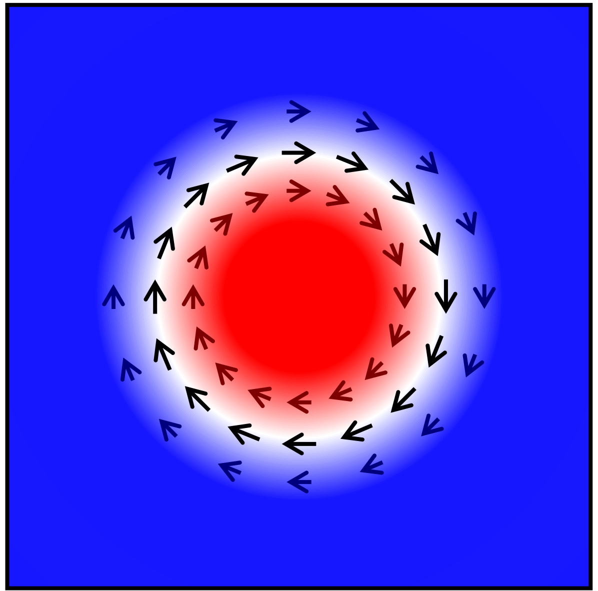


Skyrmion

$p = 1; W = 1$

$n = 1$

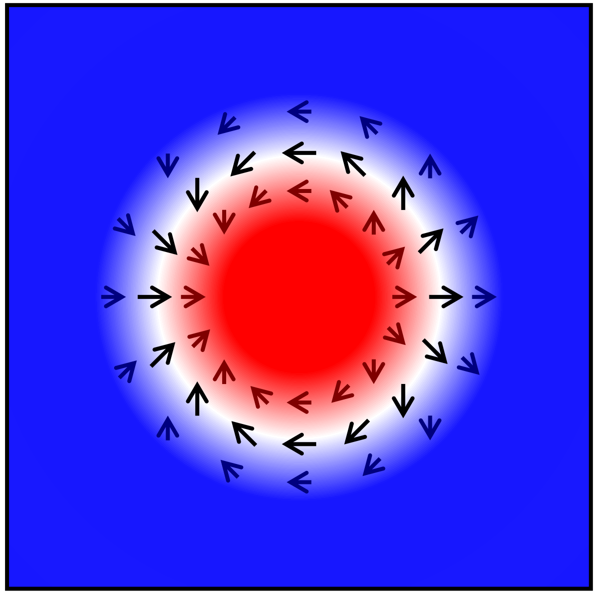




High order skyrmion:

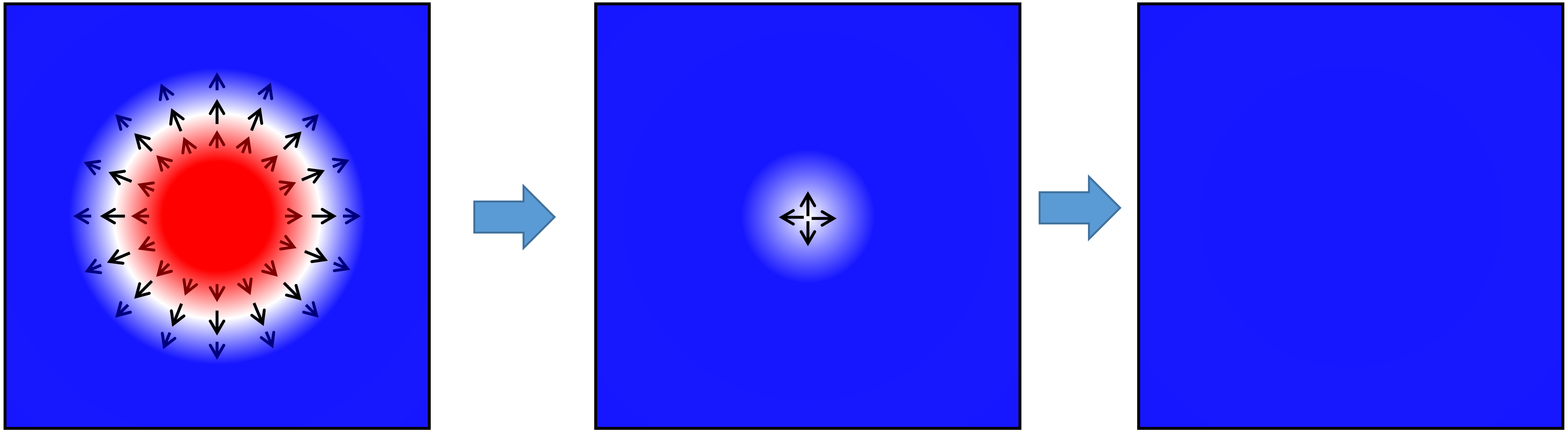
$p = 1; W = 2$

$n = 2$



$\mathbb{R}^2 \to \mathbb{S}^2$: Magnetic skyrmions and topological elements

Collapse of a skyrmion



The collapse requires a topological defect

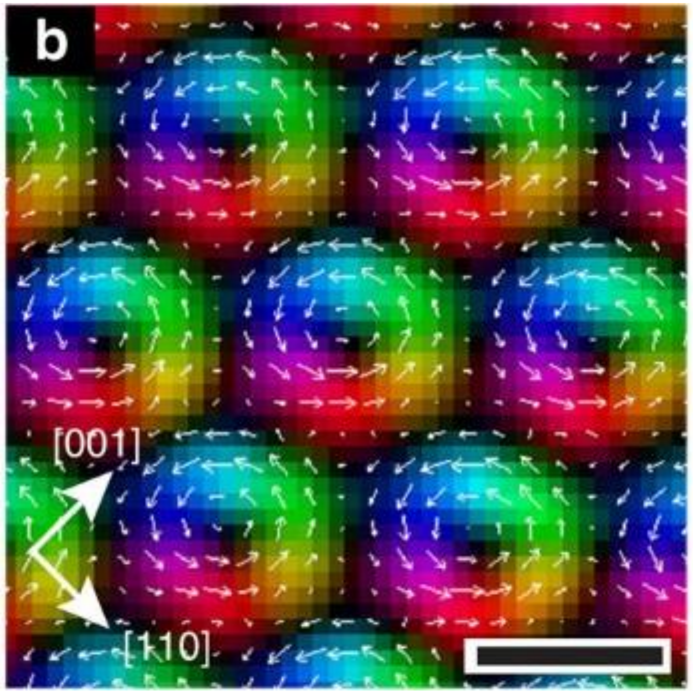
- Vortex for a 2D skyrmion

Topology changes requires to locally cancel the order parameter
$\Rightarrow$ Escape of 2D sphere $\mathbb{S}^2$ on the 2D ball $\mathbb{B}^2$ $[\pi_2(\mathbb{B}^2) = 0]$

$\mathbb{R}^2 \rightarrow \mathbb{S}^2$: Magnetic skyrmions

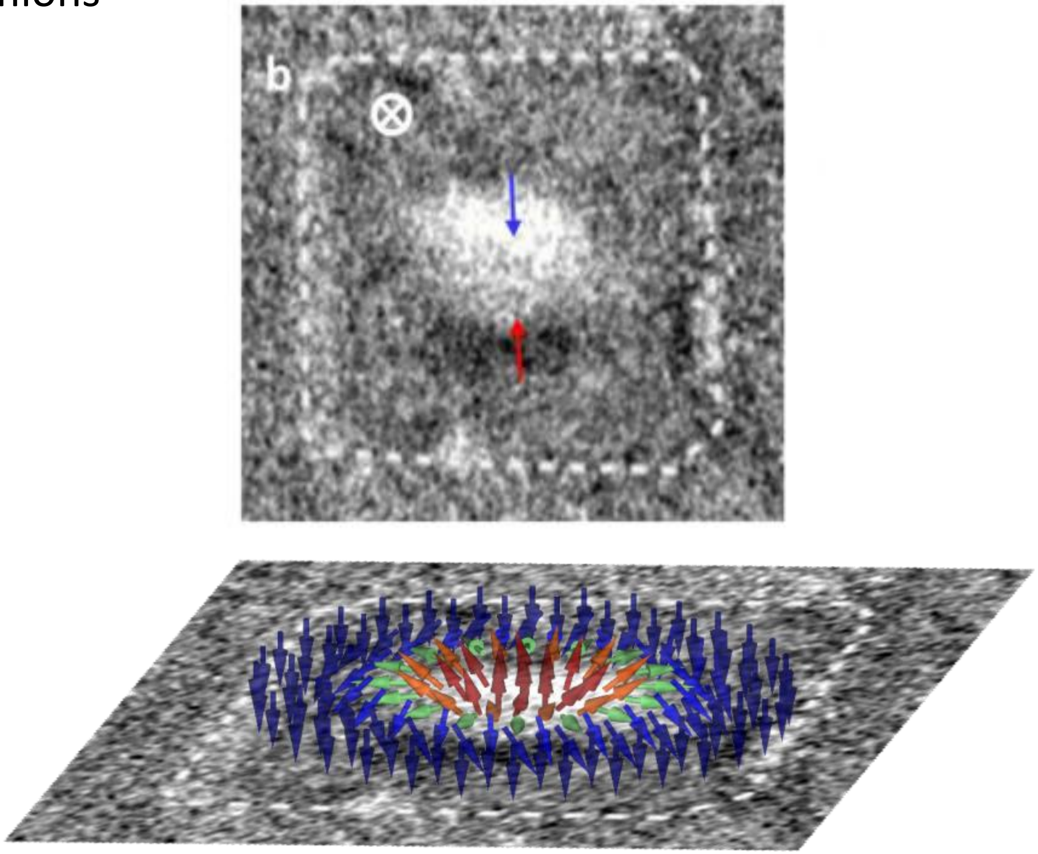
Skyrmions are obtained in situations with an absence of inversion symmetry
Dzyaloshinskii Moriya interaction fixes the chirality and therefore the topology

In Bulk crystals: Bloch skyrmions



[Yu et al. Nature (2010)]

In ultrathin films with structural absence of inversion symmetry:
Néel skyrmions



[Boule et al. Nature Nano 2016]

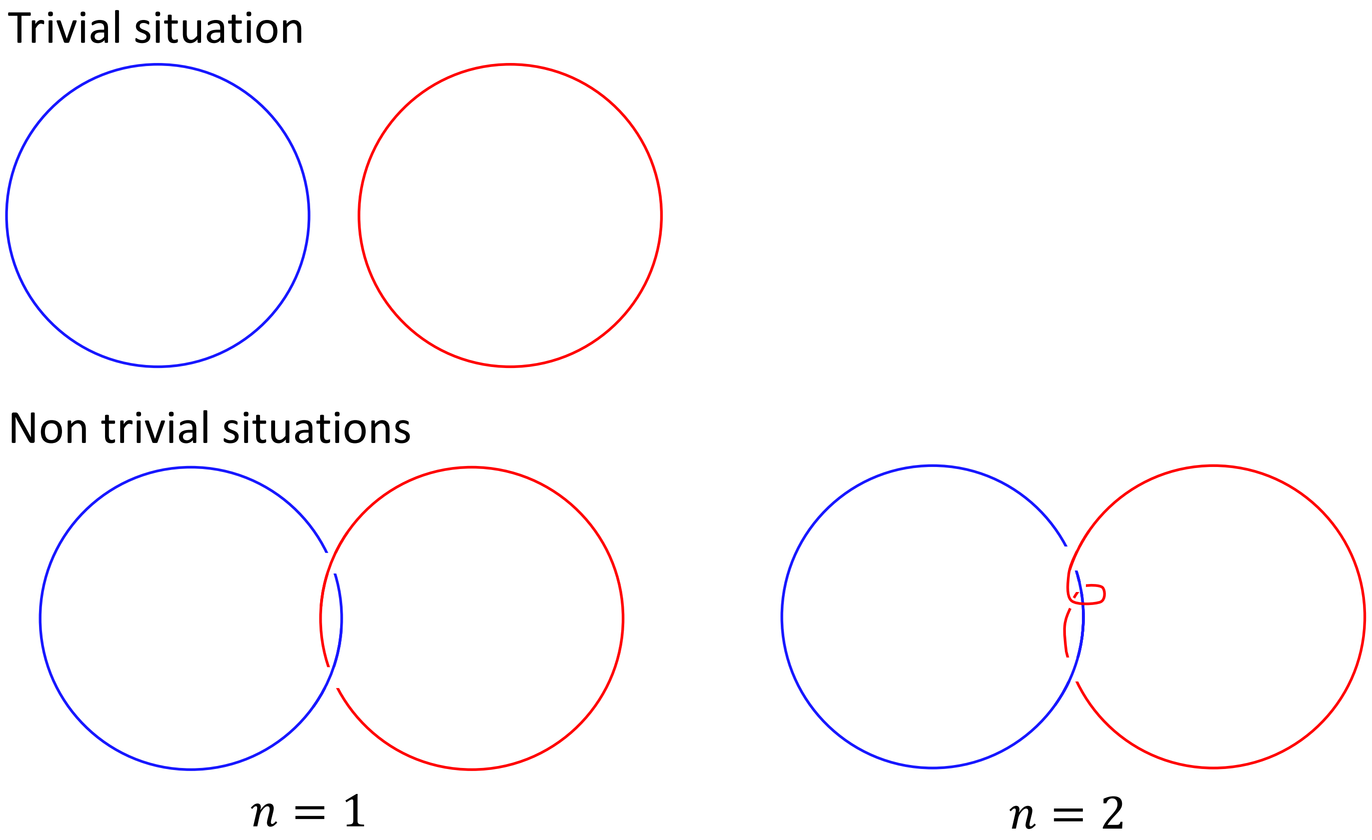
$\mathbb{R}^3 \to \mathbb{S}^2$: Hopfions

Space dimension is smaller than order parameter space:
$\Rightarrow$ Isospins are lines in $\mathbb{R}^3$.
Uniform condition at infinity
$\Rightarrow$ Isospins are circles

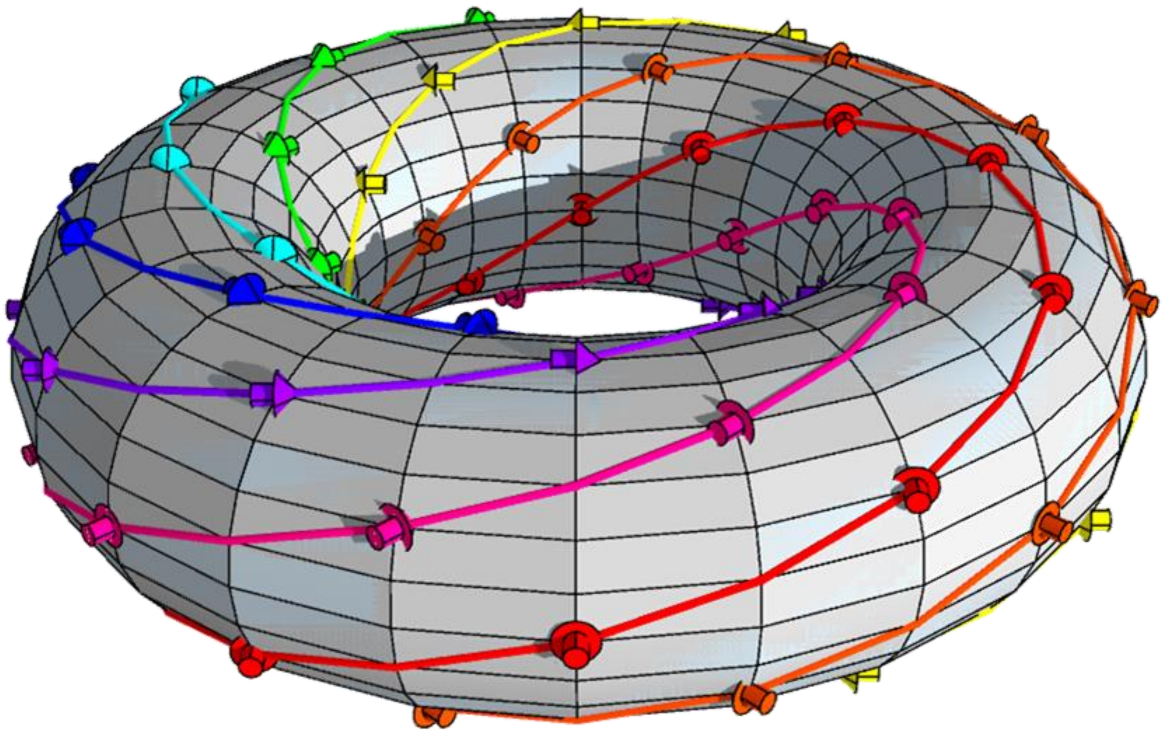
$\Rightarrow \mathbb{R}^3 \approx \mathbb{S}^2 \times \mathbb{S}^1$

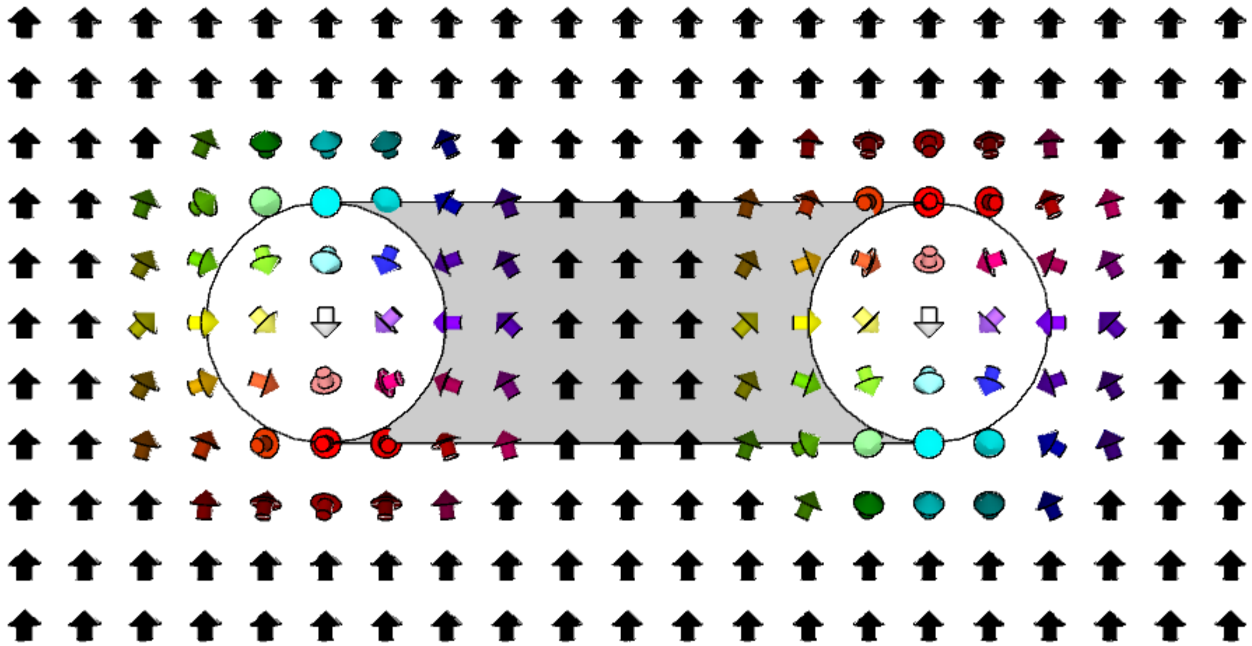
Homotopy group $\pi_3(\mathbb{S}^2)$?

$$\pi_3(\mathbb{S}^2) = \mathbb{Z}$$



$\mathbb{R}^3 \to \mathbb{S}^2$: Hopfions



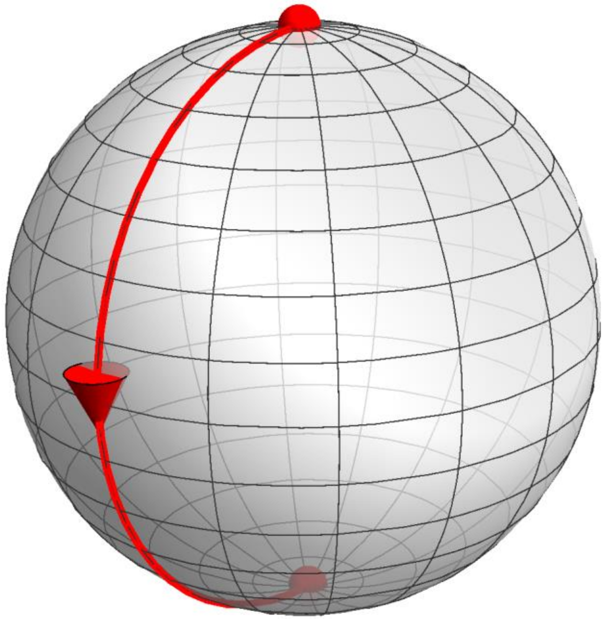


Projection of the space over the order parameter space
Case of non-uniform boundary conditions

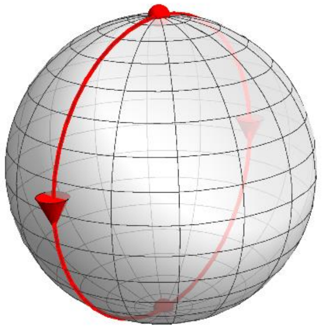
The space **partially** wraps the order parameter space
Boundary conditions are fixed by some energy => new topological solution

-1D: domain wall
Topological even with Heisenberg spins

↑↑↑↑↑↗→↘↓↓↓↓↓



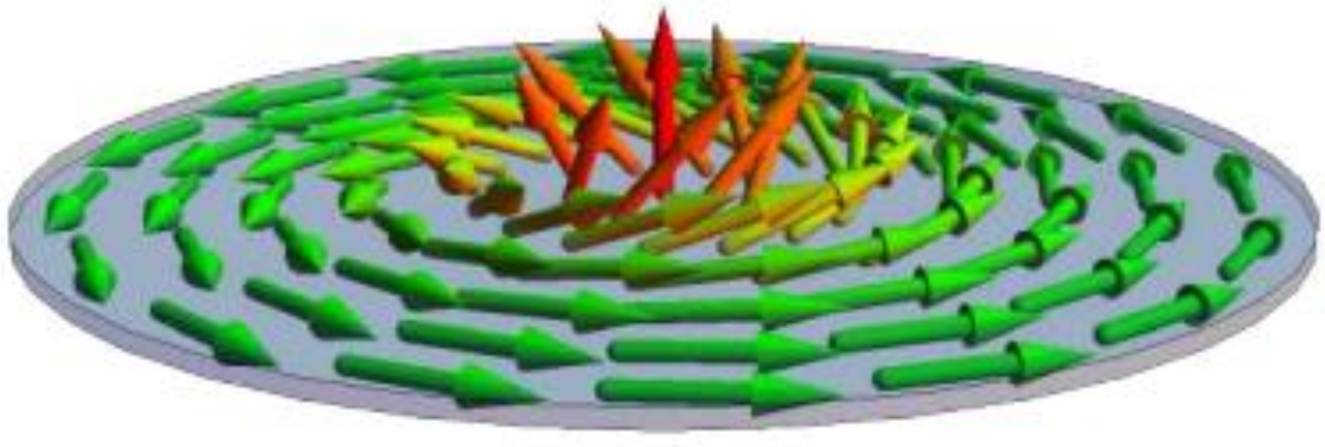
For XY spins (e.g. strong DMI, in plane anisotropy), topological difference between chiralities since magnetization in bound to a single circle $\mathbb{S}^1$

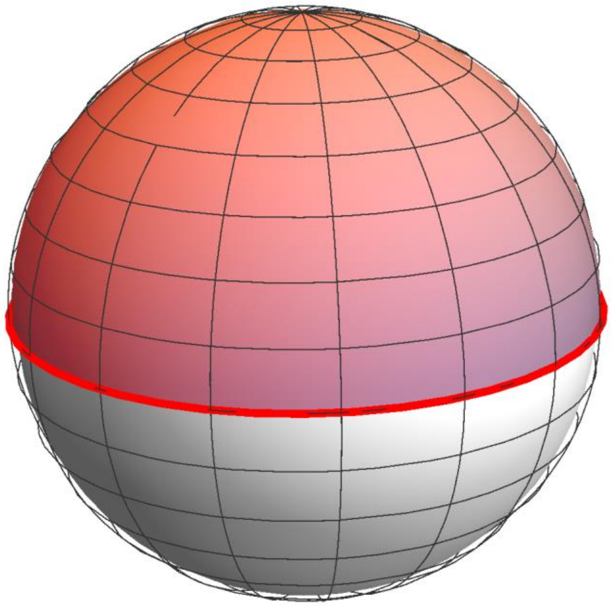


Projection of the space over the order parameter space
Case of non-uniform boundary conditions

-2D space with Heisenberg spins: micromagnetic vortex (or meron)

-> Boundary condition: $\vec{m}.\vec{n} = 0$ (magnetostatic charge minimization)
-> Dot center: $\vec{m} = \pm\vec{z}$ (exchange energy minimization)





(image from Benjamin Pigeau, Inst. Néel)

Topology depends on the vortex core orientation p and vorticity W

$$n = pW = \pm\frac{1}{2}$$

Outline

1. Introduction: homeomorphism

2. How to catch a topological defect or texture

3. Homotopy and homotopy group

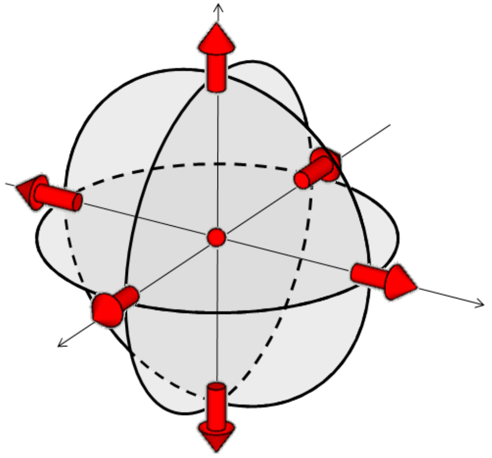
4. Geometrical space and order parameter space

5. Topological defects and topologically stable configurations

Concluding remark: topological defects and topologically stable structures

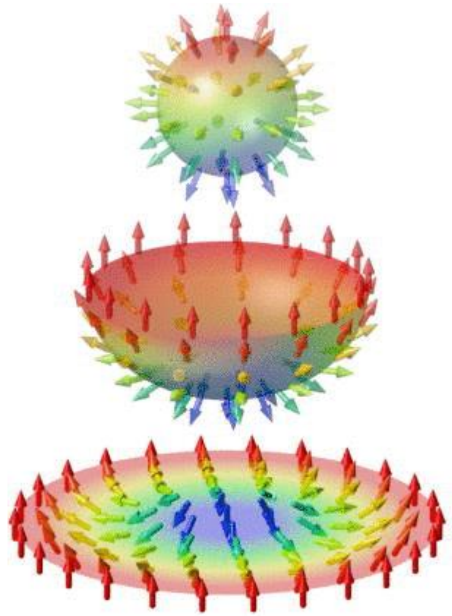
Ex: in $\pi_2(\mathbb{S}^2)$

Bloch point: topological defect



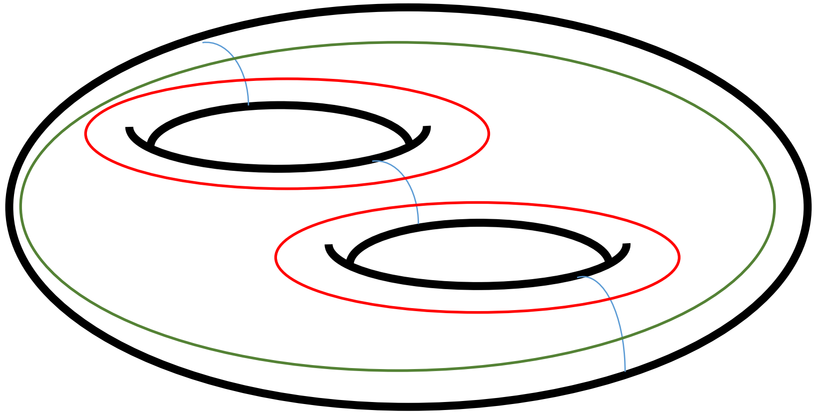
- Order parameter not defined at the defect
- Vanishing size
- Exchange energy diverges
- Stable by itself: is destroyed by a defect of opposite topological number

Skyrmion: topologically stable structure



- Order parameter always defined
- Finite size
- Finite exchange energy
- Can collapse through a topological defect

Homotopy group with n holes.



Genre 1 (tore): $\mathbb{Z}^2$
Genre 2: $\mathbb{Z}^6$
Genre 3: $\mathbb{Z}^{13}$
Genre 4: $\mathbb{Z}^{29}$
Genre 5: $\mathbb{Z}^{61}$